EXERCISES ON MOTIVES

1. Foundations

Let k be a field and S a smooth quasi-projective k-scheme. Let $\mathscr{V}_S$ be the category of smooth projective S-schemes. For an object X of $\mathscr{V}_S$ we let $\mathrm{CH}^*(X)$ be the Chow ring of X with $\mathbb{Q}$-coefficients. For X, Y in $\mathscr{V}_S$ the classes in $\mathrm{CH}^*(X \times_S Y)$ are called *correspondences* from X to Y. A correspondence $\alpha \in \mathrm{CH}^*(X \times_S Y)$ gives a group homomorphism $\alpha\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$ defined by $z \mapsto p_{Y,*}(p_X^*(z) \cdot \alpha)$, where p_X, p_Y are the projections from $X \times_S Y$ onto the two factors. Let $s\colon X \times_S Y \to Y \times_S X$ be the morphism interchanging the two factors. For any $\alpha \in \mathrm{CH}^*(X \times_S Y)$, define the *transpose* of α by ${}^t\alpha := s_*\alpha \in \mathrm{CH}^*(Y \times_S X)$.

For any morphism $f\colon X \to Y$ in $\mathscr{V}_S$, the graph morphism $\Gamma_f\colon X \to X \times_S Y$ is given by $\gamma_f(x) = (x, f(x))$ and defines the *graph correspondence* $\Gamma_f = \gamma_{f,*}[X] \in \mathrm{CH}^*(X \times_S Y)$. Define the category $\mathscr{M}_S$ of Chow motives as follows. The objects of $\mathscr{M}_S$ are triples (X, p, m) where X is an object in $\mathscr{V}_S$, $p \in \mathrm{CH}^*(X \times_S X)$ is a projector (i.e., $p \circ p = p$) and $m \in \mathbb{Z}$. For a pair of objects $(X, p, m), (Y, q, n)$ in $\mathscr{M}_S$ we define the morphisms from (X, p, m) to (Y, q, n) as follows. If $X = \coprod_j X_j$ is the decomposition of X into connected components and X_j has relative dimension $d_j := \dim(X_j/S)$ then we define

$$\mathrm{Hom}_{\mathscr{M}_S}((X, p, m), (Y, q, n)) = \{q \circ \alpha \circ p \mid \alpha \in \oplus_j \mathrm{CH}_{d_j+m-n}(X_j \times_S Y)\}$$

Take a section $e\colon S \to \mathbb{P}^1_S$ and consider the projector $[\sigma_e] = [e(S) \times_S \mathbb{P}^1_e] \in \mathrm{CH}^1(\mathbb{P}^1_S \times_S \mathbb{P}^1_S)$, which is independent of the choice of e.

Definition 1. *The motives* $1_S := (S, [S], 0)$ *and* $\mathbb{L}_S := (\mathbb{P}^1_S, [\sigma_e], 0)$ *are called the* identity motive *and the* Lefschetz motive, *respectively.*

There is a natural functor $h\colon \mathscr{V}_S \to \mathscr{M}_S$ defined on objects by $X \mapsto (X, [\Delta_{X/S}], 0)$, where $\Delta_{X/S} \subset X \times_S X$ is the diagonal. The functor h sends a morphism $f\colon X \to Y$ in $\mathscr{V}_S$ to $[\Gamma_f] \in \mathrm{Hom}_{\mathscr{M}_S}((X, \Delta_{X/S}, 0), (Y, \Delta_{Y/S}, 0))$.

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Exercise 1. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in $\mathscr{V}_S$ and let $\alpha \in \mathrm{CH}^*(X \times_S Y)$ and $\beta \in \mathrm{CH}^*(Y \times_S Z)$ be any classes. Assume that f is flat and note that g is proper because X, Y are projective over S. Show the following identities of correspondences:*

(1) $[\Gamma_g] \circ [\Gamma_f] = [\Gamma_{g \circ f}]$,
(2) $[\Gamma_g] \circ \alpha = (\mathrm{id}_X \times g)_*(\alpha)$,
(3) $\beta \circ [\Gamma_f] = (f \times \mathrm{id}_Z)^* \beta$.

Similarly, if $h\colon X \to Y$ and $j\colon Y \to Z$ are morphisms in $\mathscr{V}_S$ and h is flat then

(1) $[{}^t\Gamma_h] \circ \alpha = (\mathrm{id}_X \times h)^*(\alpha)$,
(2) $\beta \circ [{}^t\Gamma_j] = (j \times \mathrm{id}_Z)_* \beta$.

Exercise 2. *Let X be an object in $\mathscr{V}_S$ and $p \in \mathrm{CH}^*(X)$ a projector. Let $\Delta_{X/S}$ be the diagonal in $X \times_S X$. Prove:*

(1) $q := [\Delta_{X/S}] - p$ *is a projector;*
(2) $h(X) \simeq (X, p, 0) \oplus (X, q, 0)$.

Exercise 3. *Let $\mathbb{L}_S = (\mathbb{P}^1_S, [\sigma_e], 0)$ be the Lefschetz motive.*

(1) *Show that $(\mathbb{P}^1_S, [\Delta_{X/S}] - [\sigma_e], 0) \simeq (\mathbb{P}^1_S, [\mathbb{P}^1_S \times_S e(S)], 0) \simeq 1_S$. Conclude that $h(\mathbb{P}^1_S) \simeq 1_S \oplus \mathbb{L}_S$.*
(2) *More generally, prove that $h(\mathbb{P}^n_S) \simeq \bigoplus_{i=0}^n \mathbb{L}_S^{\otimes i}$.*
(3) *For any motive $M = (X, p, n) \in \mathscr{M}_S$ and $m \in \mathbb{Z}$ define $M(m) := (X, p, n + m)$. Prove that $M(-1) \simeq M \otimes \mathbb{L}_S$.*

2. Fourier transform on abelian schemes

Let $X \to S$ be an abelian scheme of relative dimension g over a smooth quasi-projective scheme S of dimension d over a field k. Let $m\colon X \times_S X \to X$ be the addition morphism $(x, y) \mapsto x + y$. Define the *Pontryagin product*

$$*\colon \mathrm{CH}^*(X) \times \mathrm{CH}^*(X) \to \mathrm{CH}^*(X)$$

by the formula $\alpha * \beta = m_*(p_1^* \alpha \cdot p_2^* \beta)$.

Exercise 4. *Check that the Pontryagin product $*$ makes $\mathrm{CH}^*(X)$ into a graded commutative ring with 1. What are the additive and multiplicative identities in this ring?*

Exercise 5. *If $f\colon X \to Y$ is a homomorphism of abelian schemes over S, show that $f_*(\alpha * \beta) = (f_*\alpha) * (f_*\beta)$ for all $\alpha, \beta \in \mathrm{CH}^*(X)$. Hint: use the identity $f \circ m_X = m_Y \circ (\mathrm{id}_Y \times f) \circ (f \times \mathrm{id}_X)$.*

Let $X^t := \mathrm{Pic}^0_{X/S}$ be the dual abelian scheme of S and let $\mathscr{P}_X$ be the Poincaré sheaf on $X \times_S X^t$. Also let $\ell := c_1(\mathscr{P}_X) \in \mathrm{CH}^1(X \times_S X^t)$ be the first Chern class of $\mathscr{P}_X$.

Definition 2. *The* Fourier transform $\mathcal{F}\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(X^t)$ *is the group homomorphism defined by the correspondence*

$$e^\ell = 1 + \ell + \frac{\ell^2}{2!} + \ldots \in \mathrm{CH}^*(X \times_S X^t).$$

Also, we let $\mathcal{F}^t\colon \mathrm{CH}^*(X^t) \to \mathrm{CH}^*(X^{tt}) = \mathrm{CH}^*(X)$ be the group homomorphism defined by the transpose of e^ℓ.

Properties. *The Fourier transform enjoys the following properties:*

(1) $\mathcal{F}([X]) = (-1)^g e^t_*[S]$, *where $e^t\colon S \to X^t$ is the zero section.*
(2) Fourier inversion: $\mathcal{F}^t \circ \mathcal{F} = (-1)^g(-\mathrm{id}_X)^*$.
(3) Exchange: $\mathcal{F}(\alpha * \beta) = \mathcal{F}(\alpha) \cdot \mathcal{F}(\beta)$ *and* $\mathcal{F}(\alpha \cdot \beta) = (-1)^g \mathcal{F}(\alpha) * \mathcal{F}(\beta)$.

If $x \in X(S)$ define $[\Gamma_x] := x_*[S] = [x(S)] \in \mathrm{CH}^g(X)$. Also, let $i_x := x \times \mathrm{id}_{X^t}\colon S \times_S X^t \to X \times_S X^t$.

Exercise 6. *Assume $d = \dim S$ and $X \to S$ has relative dimension g. For all $x \in X(S)$ prove the following formulas (originally due to Beauville):*

(1) $\mathcal{F}([\Gamma_x]) = e^{i_x^*\ell}$,
(2) $\mathcal{F}^t(i_x^*\ell) = (-1)^g \displaystyle\sum_{j=1}^{g+d} \frac{([\Gamma_e] - [\Gamma_x])^{*j}}{j}$.

Hint: $t = -\log(1 - (1 - e^{-t}))$.

3. Motivic decomposition for abelian schemes

The aim of the following batch of exercises is to prove the following motivic decomposition for abelian schemes due to Deninger and Murre.

Theorem (Deninger, Murre)**.** *Let $X \to S$ be an abelian scheme of relative dimension g over a smooth quasi-projective scheme S of dimension d over a field k. There is a* unique *decomposition*

$$[\Delta_{X/S}] = \sum_{i=0}^{2g} \pi_i \tag{3.1}$$

such that

$$\pi_i \circ \pi_j = \begin{cases} 0 & \text{if } i \neq j, \\ \pi_j & \text{if } i = j, \end{cases}$$

and such that

$$[{}^t\Gamma_n] \circ \pi_i = n^i \pi_i \text{ for all } n \in \mathbb{Z}. \tag{3.2}$$

Moreover, ${}^t\pi_i = \pi_{2g-i}$.

With the notation as above, fix a section $e\colon S \to X$. Consider the abelian scheme $p_1\colon X \times_S X \to X$, where p_1 is the first projection. For $n \in \mathbb{Z}$, let $n\colon X \to X$ denote the morphism $x \mapsto nx$ and let $[\Gamma_n] \in \mathrm{CH}^g(X \times_S X)$ be the graph correspondence of n. Note that $[\Gamma_0] = [X \times e(S)]$ and $[\Gamma_1] = [\Delta_{X/S}]$. For $i \leq 2g$ define $\pi_i \in \mathrm{CH}^*(X \times_S X)$ by

$$\pi_i := \frac{1}{(2g-i)!} \log([\Gamma_1])^{*(2g-i)} = \frac{1}{(2g-i)!} \left(\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} ([\Gamma_1] - [\Gamma_0])^{*j} \right)^{*(2g-i)}.$$

Note that for dimension reasons $\pi_i = 0$ for $i < -d$ and $\pi_{2g} = [X \times e(S)]$.

Exercise 7. *Prove the unicity part of Deninger-Murre theorem.*

Exercise 8. *Using the identity* $\exp(\log(1+x)) = 1+x$ *of formal power series prove the equality of classes*

$$[\Delta_{X/S}] = \sum_{i=-d}^{2g} \pi_i.$$

Exercise 9. *In the notation of Section 2, the Theorem of the Square states $i^*_{x+y}\ell = i^*_x\ell + i^*_y\ell$. Prove that for $x, y \in X(S)$ we have $[\Gamma_x] * [\Gamma_y] = [\Gamma_{x+y}]$.*

Exercise 10. *For any $n \in \mathbb{Z}$ and $\alpha, \beta \in \mathrm{CH}^*(X \times_S X)$ prove the identity*

$$[\Gamma_n] \circ (\alpha * \beta) = ([\Gamma_n] \circ \alpha) * ([\Gamma_n] \circ \beta).$$

Exercise 11. *Using the preceding three exercises and the equality* $\log([\Gamma_1]^{*n}) = n\log([\Gamma_1])$ *prove that for each $n \in \mathbb{Z}$ the following identity holds*

$$[\Gamma_n] \circ \pi_i = n^{2g-i} \pi_i.$$

Exercise 12. *Check the identities*

(1) $[\Gamma_n] = [\Gamma_n] \circ [\Delta_{X/S}] = \sum_{i=-d}^{2g} n^{2g-i}\pi_i$,
(2) $n^{2g-j}\pi_j = \sum_{i=-d}^{2g} n^{2g-i}\pi_i \circ \pi_j$.

Conclude that

$$\pi_i \circ \pi_j = \begin{cases} 0 & \text{if } i \neq j, \\ \pi_j & \text{if } i = j. \end{cases}$$

Exercise 13. *Using the identity* $[\Gamma_n] \circ [{}^t\Gamma_n] = n^{2g}[\Delta_{X/S}]$ *prove* ${}^t\pi_i \circ [\Gamma_n] = n^i \cdot {}^t\pi_i$ *and conclude* $[{}^t\Gamma_n] \circ \pi_i = n^i\pi_i$ *for all* $n \in \mathbb{Z}$.

Exercise 14. *Moreover,* ${}^t\pi_i = \pi_{2g-i}$ *and conclude that* $\pi_i = 0$ *for* $i < 0$.