LECTURES ON THE MMP

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1. MMP for surfaces

One of the guiding philosophies of higher dimensional geometry is that varieties are built up from one of three different pieces

Trichotomy:

(1) X is Fano, that is, $-K_X$ is ample,
(2) X is Calabi-Yau, that is, K_X is numerically trivial,
(3) X is of general type, that is, K_X is ample.

There are two aspects of this conjectural classification:

Question 1.1. *Given a variety X, how does one break it into pieces?*

Question 1.2. *How do the properties of a each piece differ?*

Fano varieties are the smallest class, and it is practical to classify restricted classes of Fano varieties. Varieties of general type are prolific, but it is quite common for them to exhibit uniform behaviour. In particular, if one fixes some numerical invariants, they form moduli spaces. There are enough Calabi-Yau's to make their classification impractical and their geometry is often surprisingly rich.

Example 1.3. *Let A be an abelian surface,*

$$A = \frac{\mathbb{C}^2}{\Lambda},$$

where $\Lambda \simeq \mathbb{Z}^2$ is lattice.

Let S be the blow up of A at the origin. Then S is not an algebraic group. Note that A does not contain any rational curves, so that points of S come in two types. Those points contained in a rational curve (namely the points of E) and points not contained in a rational curve. Thus S does not even admit a transitive action of an algebraic group.

The aim of the MMP is to undo the action of blowing up, to replace S by A. We describe the MMP for surfaces. Note that the curve E is a -1-curve,

$$E^2 = -1 \qquad K_S \cdot E = -1 \qquad \text{and} \qquad E \text{ is rational.}$$

Date: May 22, 2012.

Definition 1.4. *Let X be a normal variety and let D be an $\mathbb{R}$-Cartier divisor, a real linear combination of Cartier divisors.*

We say that D is nef if $D \cdot C \geq 0$ for all curves C.

Algorithm: MMP for surfaces

(1) Start with a smooth projective surface S.
(2) Is K_S nef? If yes, then stop.
(3) If not, then there is a curve $C \simeq \mathbb{P}^1$ such that $K_S \cdot C < 0$ and a contraction morphism $\pi\colon S \longrightarrow Z$ with one of three possibilities:
   (a) $S = \mathbb{P}^2$ and Z is a point. $-K_S$ is ample, stop.
   (b) Z is a smooth curve and $-K_S$ is relatively ample. S is a $\mathbb{P}^1$-bundle over Z, stop.
   (c) $Z = T$ is a smooth surface, π is an isomorphism outside C. In this case replace S by T and return to (2).

Remark 1.5. *The algorithm always terminates, since at each stage the Picard number of S drops by one.*

Note that the endproduct is a composed of the basic building blocks, except in the last case when K_S is nef.

Definition 1.6. *Let X be a normal variety and let D be an $\mathbb{R}$-Weil divisor, a real linear combination of Weil divisors.*

We say that D is semiample if there is a contraction morphism $\pi\colon X \longrightarrow Y$ and an ample $\mathbb{R}$-Cartier divisor H such that $D = \pi^* H$.

If D is a Cartier divisor, or even a $\mathbb{Q}$-Cartier divisor, then D is semiample if and only if there is a positive integer m such that $|mD|$ is base point free.

Theorem 1.7 (Abundance for surfaces)**.** *Let S be a smooth projective surface.*

If K_S is nef then it is semiample.

If K_S is nef then there is a contraction morphism $\pi\colon S \longrightarrow Z$ and there are three cases. If $K_S^2 > 0$ then π is birational and K_T is ample. If $K_S^2 = 0$ and Z is a smooth curve then the fibres of π are K_S-trivial, that is, they are elliptic curves. Otherwise Z is a point and K_S is numerically trivial.

Theorem 1.8 (Kodaira vanishing)**.** *Let X be a smooth projective variety and let H be an ample divisor.*

Then $H^i(X, \mathcal{O}_X(K_X + H)) = 0$.

Theorem 1.9 (Castelnuovo's contractibility criteria)**.** *Let S be a smooth projective surface and let $E \subset S$ be a -1-curve.*

Then there is a birational morphism $\pi\colon S \longrightarrow T$, T smooth, projective and π is an isomorphism outside E.

Proof. Let H be an ample divisor such that $K_S + H$ is ample. Note that $H \cdot E = k \in \mathbb{Z}$ is a positive integer. Let

$$D = K_S + H + (k-1)E.$$

Note that D is only zero on E and in fact the base locus of D is supported on E. There is a short exact sequence

$$0 \longrightarrow \mathcal{O}_S(D-E) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_E(D) \longrightarrow 0.$$

We check that

$$D - E - K_S = H + (k-2)E$$

is ample. We apply Nakai's criteria. $D - K_S - E$ is positive on E and positive on every other curve. So we just need to check that the self-intersection is positive.

$$\begin{aligned}(H + (k-2)E)^2 &= H^2 + 2(k-2)H \cdot E + (k-2)^2 E^2 \\ &= H^2 + 2(k-2)k - (k-2)^2 \\ &> 0.\end{aligned}$$

Thus

$$h^1(S, D - K_S - E) = 0,$$

by Kodaira vanishing. Since $D|_E$ is trivial and $E \simeq \mathbb{P}^1$, $D|_E$ is base point free. But then D is base point free. □

Example 1.10. *Let $\pi\colon S \longrightarrow \Delta$ be a family of curves degenerating to the union of three smooth curves, C_1, E and C_2. Suppose that C_i are irrational and E is rational. Suppose further that E is the centre of a chain, so that E meets C_1 and C_2 transversally at one point. Then $F = C_1 + E + C_2$ is not stable, we need to contract E.*

$$0 = F \cdot E = (C_1 + E + C_2) \cdot E = 1 + E^2 + 1,$$

that is $E^2 = -2$ is a -2-curve. We can contract this curve as above but if we do so then we get a singular surface.

So in the MMP we have to deal with singularities. The trichotomy above spills over to the classification of singularities.

Definition 1.11. *(X, Δ) is a **log pair**, if X is a normal variety, $\Delta \geq 0$ and $K_X + \Delta$ is $\mathbb{R}$-Cartier.*

*We say that (X, Δ) is **log smooth** if X is smooth and the support of Δ is a divisor with global normal crossings.*

Let $\pi\colon Y \longrightarrow X$ be a projective birational morphism, such that the support of the strict transform of Δ union the exceptional locus is log smooth. We may write

$$K_Y + \tilde{\Delta} + E = \pi^*(K_X + \Delta) + \sum a_i E_i,$$

where $\tilde{\Delta}$ is the strict transform of Δ, $E = \sum E_i$ is the sum of all the exceptional divisors and $a_1, a_2, \dots, a_k$ are real numbers.

*The **log discrepancy of** (X, Δ) **with respect to** E_i is a_i. The **log discrepancy** a **of** (X, Δ) is the infimum of a_i over all Y.*

There are three cases.

1. *If $a > 0$, and $\lfloor \Delta \rfloor = 0$, then we say that (X, Δ) is **kawamata log terminal**.*
2. *If $a \geq 0$, then we say that (X, Δ) is **log canonical**.*
3. *If $a < 0$, then $a = -\infty$.*

Example 1.12. *Let S be the cone over a smooth curve C of genus g. If we blow up $\pi\colon T \longrightarrow S$ the vertex of the cone, T is a smooth surface and the exceptional divisor E is a copy of C. Let $d = E^2 < 0$ be the self-intersection of E (which is nothing but the negative of the degree of C).*

We may write

$$K_T + E = \pi^* K_S + aE,$$

for a unique rational number a, the log discrepancy of S. If we dot both sides of this equation with E then we get

$$2g - 2 = (K_T + E) \cdot E = (\pi^* K_S + aE) \cdot E = aE^2 = ad.$$

It follows that

$$a = \frac{2g-2}{d}.$$

The three interesting cases break up as follows:

1. *$2g - 2 < 0$ if and only if $g = 0$ if and only if $a > 0$;*
2. *$2g - 2 = 0$ if and only if $g = 1$ if and only if $a = 0$, and*
3. *$2g - 2 > 0$ if and only if $g > 1$ if and only if $a < 0$.*

When S is a cone over a smooth curve of genus g then S is kawamata log terminal if and only if C is rational and log canonical but not kawamata log terminal if and only if C is an elliptic curve.

Note that the definition diverges a little from the guiding philosophy above. The Calabi-Yau case corresponds to the case when (X, Δ) is log canonical but not kawamata log terminal.

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