2. Finiteness of models

Question 2.1. *Given a family $\pi\colon X \longrightarrow B$, can we find a family $\psi\colon Y \longrightarrow C$ such that C is uniruled (rationally connected?) and if X_b is a Fano variety then we may find $c \in C$ such that Y_c is isomorphic to X_b.*

Roughly speaking the question asks if every component of the moduli space of Fano varieties is uniruled (or rationally connected), except that the moduli space does not really exist. For example, for del Pezzo's this is okay. E.g the del Pezzo's of degree one are obtained by blowing up eight points in $\mathbb{P}^2$ in general position, so that for C we may take an open subset of the product of eight copies of $\mathbb{P}^2$. Similarly if one goes through the list of Fano 3-folds, one can check this result case by case.

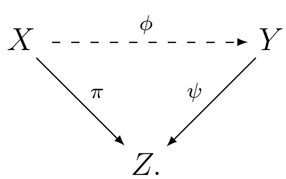
Definition 2.2. *Let X be a normal variety. We say that X is $\mathbb{Q}$-factorial if every $\mathbb{R}$-Weil divisor is $\mathbb{R}$-Cartier.*

Algorithm: MMP for varieties

(1) Start with a kawamata log terminal pair (X, Δ), where X is a $\mathbb{Q}$-factorial projective variety.
(2) Is $K_X + \Delta$ nef? If yes, then stop.
(3) If not, then there is a rational curve C such that $(K_X+\Delta)\cdot C < 0$ and a contraction morphism $\pi\colon X \longrightarrow Z$, $\pi(C)$ is a point, $\rho(X/Z) = \rho(X) - \rho(Z) = 1$, with one of three possibilities:
  (a) π is a Mori fibre space. $\dim Z < \dim X$, $-(K_X + \Delta)$ is ample over Z, stop.
  (b) π is a birational morphism and the exceptional locus is a divisor. In this case replace X by Z and return to (2).
  (c) π is a birational morphism and the exceptional locus has codimension at least two. In this case replace X by Y the flip of $\pi\colon X \longrightarrow Z$ and return to (2).

Definition 2.3. *Let $\pi\colon X \longrightarrow Z$ be a small birational morphism of relative Picard number one. Suppose that (X, Δ) is kawamata log terminal, X is $\mathbb{Q}$-factorial and $-(K_X + \Delta)$ is ample over Z.*

The flip of π is another small birational map $\psi\colon Y \longrightarrow Z$ such that $K_Y + \Gamma$ is ample over Z, where Γ is the strict transform of Δ:



Example 2.4. *Let Z be the quadric cone,*

$$(xt - yz = 0) \subset \mathbb{C}^4.$$

Note that there are two families of planes containing the origin, given by taking the cone over a line of either ruling. Both divisors are Weil divisors which are not $\mathbb{Q}$-Cartier. There are two small resolutions, $f_i \colon X_i \longrightarrow Z$, which blow up either plane. The resulting birational map $X_1 \dashrightarrow X_2$ is a flop. There is a rational curve $C_i \subset X_i$ contracted down to Z and $K_{X_i} \cdot C_i = 0$. To create a flip, simply pick up an ample divisor H_2 on X_2 and take $\Delta = \epsilon H_1$, where H_1 is the transform of H_2. Then $-(K_X + \Delta)$ is always ample over Z and if we choose $\epsilon > 0$ sufficiently small then (X, Δ) is kawamata log terminal.

Note that if we blow up the vertex of the cone then the exceptional divisor is a copy of $\mathbb{P}^1 \times \mathbb{P}^1$ and X_1 and X_2 correspond to the two rulings.

Remark 2.5. *The steps of the MMP preserve the fact that (X, Δ) is kawamata log terminal and X is $\mathbb{Q}$-factorial.*

Note that now it is no longer so obvious that the MMP always terminates. We cannot keep contracting divisors, but there might be an infinite sequence of flips:

Conjecture 2.6. *Let (X, Δ) be a kawamata log terminal pair, where X is projective, $\mathbb{Q}$-factorial.*

Then every sequence of flips terminates.

In terms of our original goal, even if know (2.6), we still need:

Conjecture 2.7 (Abundance conjecture)**.** *Let (X, Δ) be a kawamata log terminal pair, where X is a projective variety.*

If $K_X + \Delta$ is nef then it is semiample.

The major sticking point is

Conjecture 2.8. *Let X be a smooth projective variety.*

Either

(1) X is uniruled, or
(2) the Kodaira dimension is not $-\infty$.

Definition 2.9. *Let X be a normal projective variety and let D be an $\mathbb{R}$-Weil divisor.*

The section ring is the graded ring

$$R(X, D) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(\lfloor mD \rfloor)).$$

Suppose that the section ring is a finitely generated $\mathbb{C}$-algebra. Then there is a rational map

$$f_D \colon X \dashrightarrow Y$$

where $Y = \operatorname{Proj} R(X, D)$. *$Y$ is the image of X under some multiple of D,*

$$\phi_{mD} \colon X \dashrightarrow Y \subset \mathbb{P}^N.$$

If $D = K_X$ is big then Y is the canonical model. K_Y is ample and Y has canonical singularities, that is, the log discrepancy is at least one. One way to construct the canonical model, is to run the K_X-MMP until K_X is nef. At this point the base point free theorem says that K_X is semiample and Y is the image of ϕ_{mK_X}, for m sufficiently large. Note that if X_1 and X_2 are of general type then X_1 and X_2 are birational if and only if the canonical models Y_1 and Y_2 are isomorphic. However note that Y is in general note $\mathbb{Q}$-factorial.

More generally, one can work with kawamata log terminal pairs (X, Δ), where $K_X + \Delta$ is big. In this case (Y, Γ) is kawamata log terminal and $K_Y + \Gamma$ is ample. The resulting map is called the log canonical model. If the canonical ring is finitely generated then this construction makes sense even if $K_X + \Delta$ is pseudo-effective.

Theorem 2.10. *Suppose that $(X, A + D = A + \sum D_i)$ is log smooth, where X is a projective variety and A is an ample $\mathbb{Q}$-divisor.*

Then there are finitely many rational maps $f_1, f_2, \dots, f_k$, $f_i \colon X \dashrightarrow Y_i$, such that if $f \colon X \dashrightarrow Y$ is the log canonical model of $(X, \Delta = A + \sum a_i D_i)$ then $f = f_i$. Moreover the closure $\mathcal{P}_i$ of the set

$$\{\, (a_1, a_2, \dots, a_k) \in [0,1]^k \mid f_i \textit{ is the log canonical model of } (X, \Delta = \sum a_i \Delta_i) \,\},$$

is a rational polytope.

More generally one can work with a normal $\mathbb{Q}$-factorial variety X and finitely many divisors Δ_i such that (X, Δ_i) is kawamata log terminal. Also one can go relatively easily from finiteness of log canonical models to finiteness of minimal models.

Example 2.11. *Let $X = \overline{M}_g$ and let $D = \sum D_i$, the standard boundary divisors. In this case various regions of the box*

$$[0,1]^k \qquad \textit{where} \qquad k = \lfloor \frac{g}{2} \rfloor,$$

are decomposed into finitely many rational polytopes.

What does regions mean? Well to apply (2.10) *one needs to peel off an ample $\mathbb{Q}$-divisor A. Note that we are allowed to use as small an ample $\mathbb{Q}$-divisor as want.*

By an old result of Mumford, $K_X + D$ is ample on $\overline{M}_g$. So one can play a game and write

$$K_X + \Delta = a(K_X + D) + b(K_X + \Delta') = b(K_X + A + \Delta'),$$

where $A \sim_{\mathbb{Q}} a/b(K_X + \Delta)$ and $0 \leq \Delta' \leq \Delta$. If we want to fix A, that is, if we want to fix a and provided $\Delta > \delta D$, for some fixed $\delta > 0$.

In other words, if there are infinitely many log canonical models, then they can only accumulate where at least one coefficient of the boundary is zero.

But suppose that $g \geq 22$. Then K_X is big and if $\Delta \leq (1 - \epsilon)D$ then we can write

$$K_X + \Delta = \eta K_X + (1 - \eta)(K_X + \Delta'),$$

where $\Delta \leq \Delta' \leq D$. If we fix $\eta > 0$ then we are again done.

In general we play similar games to prove (2.10). One key fact is that $[0, 1]^k$ is compact, so it is enough to prove finiteness locally and then invoke compactness.

It is interesting to note that we need the ample divisor.

Example 2.12 (Reid)**.** *Let $X_0 \subset \mathbb{C}^4$ be the smooth threefold given by the equation*

$$y^2 = ((x - a)^2 - t_1)((x - b)^2 - t_2),$$

where x, y, t_1, t_2 are coordinates on $\mathbb{C}^4$ and $a \neq b$ are constants. Let X be the closure of X_0 in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{C}^2$. Projection $\pi\colon X \longrightarrow S = \mathbb{C}^2$ down to $\mathbb{C}^2$ with coordinates t_1 and t_2 realises X as a family of projective curves of genus one over $\mathbb{C}^2$. If $t_1 t_2 \neq 0$ then we have a smooth curve of genus one, that is, an elliptic curve. If $t_1 = 0$ and $t_2 \neq 0$ or $t_2 = 0$ and $t_1 \neq 0$ then we get a nodal rational curve (a copy of $\mathbb{P}^1$ with two points identified). If $t_1 = t_2 = 0$ then we get a pair $C_1 \cup C_2$ of copies of $\mathbb{P}^1$ joined at two points.

One can check that both C_1 and C_2 can be contracted individually to a simple node. Therefore we can flop either C_1 or C_2. Suppose that we flop C_1. Since C_1 is contracted by π this flop is over S so that the resulting threefold Y admits a morphism to $\psi\colon Y \longrightarrow S = \mathbb{C}^2$. We haven't changed the morphism π outside s and one can check that the fibre over $s = (0, 0)$ of ψ is a union $D_1 \cup D_2$ of two copies of $\mathbb{P}^1$ which intersect in two different points. Once again we can flop either of these curves. Suppose that D_2 is the strict transform of C_2 so that D_1 is the flopped curve. If we flop D_1 then we get back to X but if we flop D_2 then we get another threefold which fibres over S. Continuing in this way we get infinitely many threefolds all of which admit a morphism to S and all of which are isomorphic over the open set $S - \{s\}$. Let G

be the graph whose vertices are these threefolds, where we connect two vertices by an edge if there is a flop between the two threefolds over S. Let G' be the graph whose vertices are the integers where we connect two vertices i and j if and only if $|i-j|=1$. Then G and G' are isomorphic.

There are examples due to Kawamata of Calabi-Yau threefolds with infinitely many models.

Algorithm: MMP with scaling

(1) Start with a kawamata log terminal pair $(X, \Delta + D)$, where X is a $\mathbb{Q}$-factorial projective variety, Δ is big, such that $K_X + \Delta + D$ is nef.

(2) Let

$$\lambda = \inf\{\, t \in [0, 1] \,|\, K_X + \Delta + tD \text{ is nef.} \,\}$$

Is $\lambda = 0$? If yes, then $K_X + \Delta$ is nef, stop.

(3) If not, then there is a rational curve C such that $D \cdot C > 0$, $(K_X + \Delta + \lambda D) \cdot C = 0$ and a contraction morphism $\pi \colon X \longrightarrow Z$, $\pi(C)$ is a point, $\rho(X/Z) = \rho(X) - \rho(Z) = 1$, with one of three possibilities:
   (a) π is a Mori fibre space. $\dim Z < \dim X$, $-(K_X + \Delta)$ is ample over Z, stop.
   (b) π is a birational morphism and the exceptional locus is a divisor. In this case replace X by Z and return to (2).
   (c) π is a birational morphism and the exceptional locus has codimension at least two. In this case replace X by Y the flip of $\pi \colon X \longrightarrow Z$ and return to (2).

Note that the MMP with scaling always terminates, using finiteness of minimal models. Indeed, every step of this MMP we get a minimal model