3. Variations on a theme

There are some interesting variations on a theme. One possibility is to run a relative MMP. Here we start with $X \longrightarrow W$. We only contract curves contracted over W. Thus every step is over W. Big over W means big on the general fibre. So everything is big if $X \longrightarrow W$ is birational.

For the next result we will need the very standard:

Lemma 3.1 (Negativity Lemma)**.** *Let $\pi\colon X \longrightarrow Y$ be a birational projective morphism between normal varieties.*

If E is any exceptional divisor such that $-E$ is nef, then $E \geq 0$. In particular if $E \geq 0$ is exceptional then E is nef if and only if $E = 0$.

Theorem 3.2. *Let (W, Θ) be a kawamata log terminal pair.*

Then there is a small birational morphism $\pi\colon X \longrightarrow W$ such that if Δ is the strict transform of Θ, then (X, Δ) is kawamata log terminal and X is $\mathbb{Q}$-factorial.

Proof. As a first approximation to π, take a log resolution. By assumption if we write

$$K_X + \tilde{\Theta} + E = \pi^*(K_W + \Theta) + \sum a_i E_i,$$

then every $a_i > 0$. Let

$$\Delta = \tilde{\Theta} + (1 - \epsilon)E,$$

where $\epsilon >$ is sufficiently small ($a/2$ will do, where a is the log discrepancy). Then

$$K_X + \Delta = \pi^*(K_W + \Theta) + F,$$

where $F \geq 0$ is exceptional.

We want to contract the exceptional divisors, which by construction coincide with the components of F. Pick D ample over W so that $K_X + \Delta + D$ is ample over W. We run the $(K_X + \Delta)$-MMP with scaling of D over W.

Since we work over W, $K_X + \Delta$ is automatically big over W. So the MMP terminates. It cannot terminate with a Mori fibre space, since Z sits between X and W. Thus at the end $K_X + \Delta$ is nef.

The key point is negativity of contraction. $K_X + \Delta$ is numerically equivalent over W to F. Negativity of contraction implies that $F = 0$. But then π is small. As the MMP preserves $\mathbb{Q}$-factoriality, X is $\mathbb{Q}$-factorial automatically. □

We have already seen that if one drops the existence of the ample divisor A then finiteness of models breaks down. However there is a remarkable conjecture due to Kawamata and Morrison:

Conjecture 3.3. *Suppose that $K_X + \Delta$ is kawamata log terminal and numerically trivial.*

Then there are only finitely many minimal models, up to the action of the birational automorphism group.

We have already seen an example of this in the relative setting. In Reid's example, there are infinitely many minimal models and yet they are all isomorphic. There are also interesting examples due to Hassett and Tschinkel, constructed using cubic fourfolds.

Definition 3.4. *Let X be a smooth projective variety. Let $D_1, D_2, \dots, D_k$ be a sequence of $\mathbb{R}$-divisors. The* ***multi-graded section ring*** *associated to $D_1, D_2, \dots, D_k$ is the ring*

$$R(X, D_1, D_2, \dots, D_k) = \bigoplus_{m \in \mathbb{N}^k} H^0(X, \mathcal{O}_X(\lfloor D \rfloor)) \quad \text{where} \quad D = \sum m_i D_i.$$

There is a general theory of such multi-graded rings, initiated by Hu and Keel, which states that whenever we have finite generation, we get finiteness of models.

Theorem 3.5. *Let X be a normal projective variety and let $\Delta_1, \Delta_2, \dots, \Delta_k$ be a sequence of divisors such that (X, Δ_i) is kawamata log terminal. Let $D_i = K_X + \Delta_i$.*

If Δ_i is big for $1 \leq i \leq k$ then the multigraded section ring

$$R(X, D_1, D_2, \dots, D_k)$$

is finitely generated.

Cascini and Lazić have completely reversed the logic of finite generation by giving a direct proof of (3.5) and then used these ideas to prove the standard results of the MMP (cone theorem, base point free theorem, MMP with scaling and so on).

As an easy example of this note the following:

Theorem 3.6. *Let $(X, A + \sum D_i)$ be a log smooth pair, where A is an ample $\mathbb{Q}$-divisor.*

Then the set

$$\mathcal{P} = \{\, (a_1, a_2, \dots, a_k) \in [0, 1]^k \mid K_X + A + \sum a_i D_i \sim_{\mathbb{R}} B \geq 0 \,\},$$

is a rational polytope.

Proof. Pick generators for $f_1, f_2, \dots, f_k$ for the multigraded ring. $f_i \in H^0(X, \mathcal{O}_X(\sum_j \lfloor m_{ij} D_j \rfloor))$. The integral vectors $\vec{m}_i = (m_{i1}, m_{i2}, \dots, m_{ik})$ generate the monoid of all effective integral linear combinations of $K_X + A + D_i$. The same vectors then span the cone $\mathcal{Q}$ of all pseudo-effective divisors which are convex linear combinations of $K_X + A + D_i$.

Hence $\mathcal{Q}$ is a rational polyhedron. As $\mathcal{P} = \mathcal{Q} \cap [0,1]^k$ it follows that $\mathcal{P}$ is a rational polytope. $\square$

It is interesting to note that if we try to weaken the hypothesis just a little we get two very hard conjectures:

Conjecture 3.7. *Let (X, Δ) be a log smooth log canonical pair.*

$R(X, K_X + \Delta)$ is finitely generated.

Conjecture 3.8. *Let X be a normal projective variety and let $\Delta_1, \Delta_2, \dots, \Delta_k$ be a sequence of divisors such that (X, Δ_i) is kawamata log terminal. Let $D_i = K_X + \Delta_i$.*

The multigraded section ring

$$R(X, D_1, D_2, \dots, D_k)$$

is finitely generated.

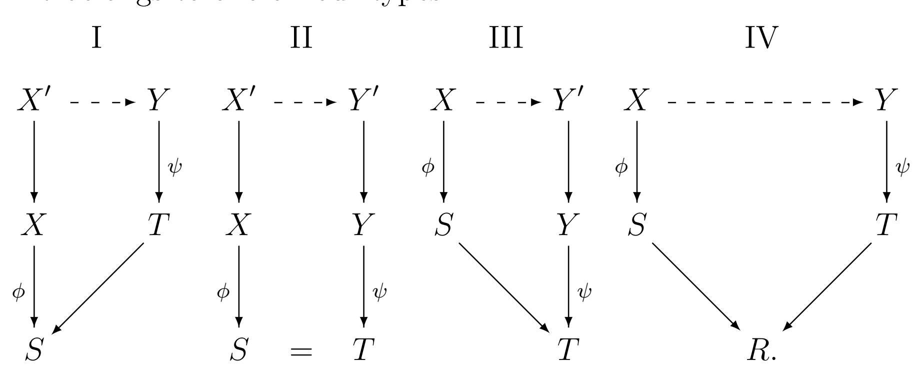
Both conjectures imply abundance and termination of flips (at least for the MMP with scaling).

Let me end with an application of the ideas behind finiteness of models to the Sarkisov program. Consider all Mori fibre spaces which are rational surfaces. They are $\mathbb{P}^2 \longrightarrow Z$ where Z is a point and $\mathbb{F}_n \longrightarrow \mathbb{P}^1$, for $n \in \mathbb{N}$, the Hirzerbruch surface, so that there are infinitely many. Nevertheless they are connected to each other in a very simple way. If one blows up $\mathbb{P}^2$ one gets $\mathbb{F}_1$ and one can get from $\mathbb{F}_n$ to $\mathbb{F}_{n\pm1}$ using an elementary transformation, blow up a fibre and blow down the old fibre. Sarkisov realised that this generalises to higher dimensions.

Theorem 3.9. *Suppose that $\phi\colon X \longrightarrow S$ and $\psi\colon Y \longrightarrow T$ are two Mori fibre spaces with $\mathbb{Q}$-factorial terminal singularities.*

Then X and Y are birational if and only if they are connected by a sequence of Sarkisov links.

A Sarkisov link relates two Mori fibre spaces $\phi\colon X \longrightarrow S$ and $\psi\colon Y \longrightarrow T$. It belongs to one of four types:



All horizontal maps are a sequence of flops (but not with respect to $K_X + \Delta$). The other maps are extremal (relative Picard number one). Maps from primes are always extremal divisorial contractions. I and III are simply reflections of each other.

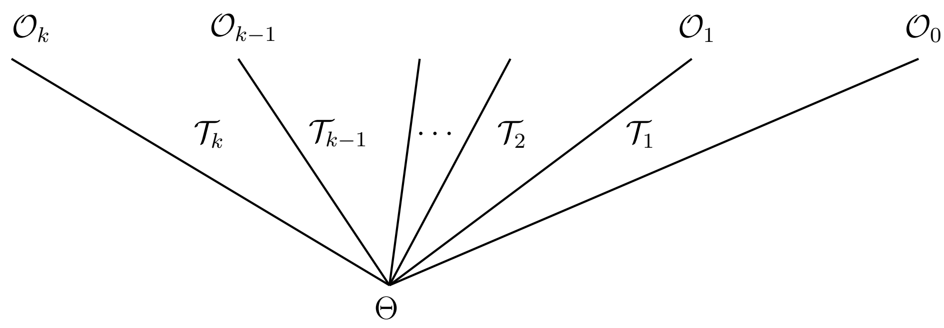
It is probably easiest to see what is going on in a specific example. One can decompose the Cremona transformation

$$\sigma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \qquad \text{where} \qquad [X : Y : Z] \longrightarrow [X^{-1} : Y^{-1} : Z^{-1}].$$

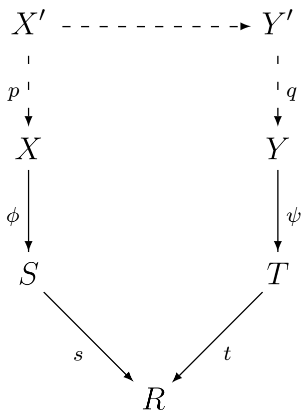
We show how to use finiteness of models to deduce (3.9). Pick sufficiently ample divisors H' and G' on S and T. Then $A \sim_{\mathbb{Q}} -(K_X + \Delta) + \pi^* H'$ and $B \sim_{\mathbb{Q}} -(K_Y + \Gamma) + \psi^* G'$ are ample. We may pick A and B so that $K_X + \Delta + A$ and $K_Y + \Gamma + B$ are kawamata log terminal.

Pick a log resolution $fW \dashrightarrow X$ and $g\colon W \longrightarrow Y$ of everything in sight. We may find C an ample $\mathbb{Q}$-divisor such that f and g are log terminal models of $K_W + C + D$ and $K_W + C + G$ for D and G supported on some log smooth divisor, and S and T are ample models. We get finiteness of models for $(W, C+G+H)$, which lives in a big vector space. Pick a general plane which contains the polytopes corresponding to X and Y. The effective cone is a rational polytope. Imagine going from X to Y along the internal boundary.

At finitely many points one sees a wedge of models:



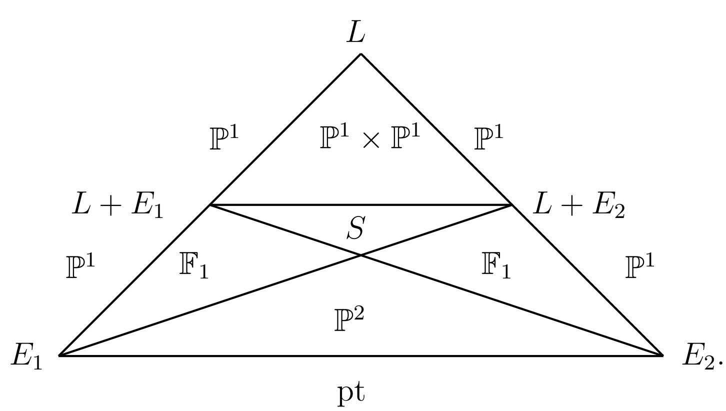
The internal boundary corresponds to the possible base of a Mori fibre space. If one polytope is contained in another this means there is a morphism between the corresponding models. The picture around Θ corresponds to an octagon of models. But the drop in the Picard number is two. So one of the three sides of each octagon is contracted; this gives the four possible types of Sarkisov link.



To illustrate some of these ideas, let us consider an easy case. Let S be the blow up of $\mathbb{P}^2$ at two points. Then S is a toric surface and there are five invariant divisors. The two exceptional divisors, E_1 and E_2, the strict transform L of the line which meets E_1 and E_2, and finally the strict transform L_1 and L_2 of two lines, one of which meets E_1 and one of which meets E_2. Then the cone of effective divisors is spanned by the invariant divisors and according to Hu and Keel the polytopes we are looking for are obtained by considering the chamber decomposition given by the invariant divisors. Since $L_1 = L + E_1$ and $L_2 = L + E_2$ the cone of effective divisors is spanned by L, E_1 and E_2. Since $-K_S$ is ample, we can pick an ample $\mathbb{Q}$-divisor A such that $K_S + A \sim_{\mathbb{Q}} 0$ and $K_S + A + E_1 + E_2 + L$ is divisorially log terminal. Let V be the real vector space of Weil divisors spanned by E_1, E_2 and L. In this case projecting

$$\mathcal{P}(V) = \{\, \Theta \,|\, \Theta = A + t_1E_1 + t_2E_2 + t_3L,\ 0 \leq t_i \leq 1 \,\}$$

from the origin we get



We have labelled each polytope by the corresponding model. Imagine going around the boundary clockwise, starting just before the point corresponding to L. The point L corresponds to a Sarkisov link of type IV_m, the point $L+E_2$ a link of type II, the point E_2 a link of type III, the point E_1 a link of type I and the point $L+E_1$ another link of type II.