Morphisms of complex tori and abelian varieties

Exercise 1. Homomorphisms of complex tori. Let $X_1 = V_1/\Lambda_1$ and $X_2 = V_2/\Lambda_2$ be two complex tori (V_i are complex vector spaces and $\Lambda_i \subset V_i$ are lattices, i.e. discrete subgroups of maximal rank). Let $f : X_1 \to X_2$ be a holomorphic map.

(a) Show that there exists an affine map $\tilde{f} : V_1 \to V_2$, that induces f. (**Hint:** use Liouville Theorem).

(b) The composition $h := t_{-f(0)} \circ f$ is a homomorphism of groups ($t_{x_0}(x) = x + x_0$ is the translation). The injective homomorphism $\operatorname{Hom}(X_1, X_2) \to \operatorname{Hom}_{\mathbb{C}}(V_1, V_2)$ which sends $h \mapsto \tilde{h}$ is called the analytic representation of $\operatorname{Hom}(X_1, X_2)$.

(c) If we consider $h : X_1 \to X_2$, then $h(X_1)$ is a subtorus of X_2 and the connected component $(\ker h)^0$ passing through 0 of $\ker h$ is a subtorus of X_1. $\ker h/(\ker h)^0$ is a finite group and $\dim X_1 = \dim(\ker h)^0 + \dim \operatorname{im} h$.

(d) Show that $\operatorname{Hom}(X_1, X_2) \cong \mathbb{Z}^m$ for some $m \leqslant 4 \dim X_1 \cdot \dim X_2$. (**Hint:** use the rational representation $\operatorname{Hom}(X_1, X_2) \to \operatorname{Hom}_{\mathbb{Z}}(\Lambda_1, \Lambda_2)$).

Line bundles on complex tori

Exercise 2. Factors of automorphy and line bundles on complex tori. Let $X = V/\Lambda$ be a complex torus. A holomorphic map $f : \Lambda \times V \to \mathbb{C}^*$ satisfying $f(\lambda + \mu, v) = f(\lambda, v + \mu) f(\mu, v)$ is called a factor of automorphy. Given an automorphy factor f we can define the following action of Λ on $V \oplus \mathbb{C}$,

$$\Lambda \ni \lambda : (v, t) \mapsto (v + \lambda, f(\lambda, v) \cdot t)$$

The quotient $L = (V \oplus \mathbb{C})/\Lambda$ is well-defined, and it is a line bundle over X. (This is a particular case of the isomorphism $H^1(\pi_1(X), H^0(\tilde{X}, \mathcal{O}^*_{\tilde{X}})) \to \ker(H^1(X, \mathcal{O}^*_X) \xrightarrow{\pi^*} H^1(\tilde{X}, \mathcal{O}^*_{\tilde{X}}))$, where $\pi : \tilde{X} \to X$ is the universal cover of X.)

(a) Let $h : V \to \mathbb{C}^*$ be a holomorphic function. Show that $h(v + \lambda) h(v)^{-1}$ is a factor of automorphy that defines the trivial line bundle on X (the cycles of this type are called coboundaries).

(b) Show that $H^2(X, \mathbb{Z}) \cong \operatorname{Alt}^2(\Lambda, \mathbb{Z})$ = group of $\mathbb{Z}$-valued alternating 2-forms on Λ. (**Hint:** Use Künneth formula).

Consider the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}^*_X \to 0$. One one hand, it gives us the following exact sequence $0 \to \mathbb{Z} \to H^0(V, \pi^*\mathcal{O}_X) \xrightarrow{e^{2\pi i}} H^0(V, \pi^*\mathcal{O}^*_X) \to 0$. On the other hand, if we consider the coboundary maps of the cohomological long exact sequences, they are compatible through the following commutative diagram:

$$\begin{array}{ccc}
H^1(\Lambda, H^0(V, \pi^*\mathcal{O}^*_X)) & \xrightarrow{\quad\delta\quad} & H^2(\Lambda, \mathbb{Z}) \\
\cong\downarrow & & \cong\downarrow \\
\operatorname{Pic} X \cong H^1(X, \mathcal{O}^*_X) & \xrightarrow{\quad\delta = c_1\quad} & H^2(X, \mathbb{Z}) \cong \operatorname{Alt}^2(\Lambda, \mathbb{Z})
\end{array}$$

Therefore, given a factor of automorphy $f = e^{2\pi i g}$ defining $L \in \operatorname{Pic} X$, the first Chern class $c_1(L)$ can be described in $\operatorname{Alt}^2(\Lambda, \mathbb{Z})$ as

$$E_L(\lambda, \mu) = g(\mu, v + \lambda) + g(\lambda, v) - g(\lambda, v + \mu) - g(\mu, v) \qquad \text{for all } v \in V.$$

(c) Show that E_L is well-defined, i.e., the definition does not depend on $v \in V$.

(d) Show that $E_L(\Lambda, \Lambda) \subseteq \mathbb{Z}$ and after extending E_L $\mathbb{R}$-linearly, we get $E_L(iv, iw) = E_L(v, w)$ for all $v, w \in V$. (**Hint:** Use that the map $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$ factorizes through $H^2(X, \mathbb{C})$, the Hodge decomposition and the isomorphism $H^q(X, \Omega^p_X) \cong \wedge^p T \otimes \wedge^q \overline{T}$, where $T = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.)

(e) There is a 1-1 correspondence between the set of hermitian forms H on V and the set of real valued alternating forms E on V satisfying $E(iv, iw) = E(v, w)$. (**Note:** With the convention $H(v, w) = E(iv, w) + iE(v, w)$, the hermitian forms become holomorphic on the first factor).

(f) Show that E is non-degenerate if, and only if, H is non-degenerate.

Recall that the image of c_1 is called the Neron-Severi group $\mathrm{NS}\,X$ of X. A semicharacter for an hermitian form $H \in \mathrm{NS}\,X$ is a map $\chi : \Lambda \to U(1) = \{z \in \mathbb{C} \,|\, |z| = 1\}$ such that

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{i\pi \mathrm{Im}\, H(\lambda, \mu)},$$

so when $H = 0$ it is a character.

(g) Given (H, χ), where χ is a semicharacter (s.c.) for $H \in \mathrm{NS}\,X$.Show that

$$a_{(H,\chi)}(\lambda, v) := \chi(\lambda)e^{\pi H(v,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)}$$

is a factor of automorphy (it is usually called canonical factor of (H, χ)).

(h) Check that the following diagram commutes:

$$\begin{array}{ccc}
\{(H,\chi)\}_{\chi \text{ s.c. of } H \in \mathrm{NS}\,X} & \xrightarrow{p_1} & \mathrm{NS}\,X \\
\downarrow{\scriptstyle \psi} & & \| \\
\mathrm{Pic}\,X & \xrightarrow{c_1} & \mathrm{NS}\,X,
\end{array}$$

where $\psi(H, \chi) = a_{(H,\chi)}$.

(h) Finally to show the Appell-Humbert theorem, i.e. the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(\Lambda, U(1)) & \longrightarrow & \{(H,\chi)\}_{\chi \text{ s.c. of } H \in \mathrm{NS}\,X} & \xrightarrow{p_1} & \mathrm{NS}\,X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \| & & \\
0 & \longrightarrow & \mathrm{Pic}^0\,X & \longrightarrow & \mathrm{Pic}\,X & \xrightarrow{c_1} & \mathrm{NS}\,X & \longrightarrow & 0,
\end{array}$$

we need to show that ϕ is an isomorphism (**Hint:** Consider $\mathrm{Pic}^0\,X \cong \mathrm{Im}\,(H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}^*_X)) \cong \mathrm{Im}\,(H^1(X, \mathbb{C}) \xrightarrow{\varepsilon} H^1(X, \mathcal{O}^*_X))$, where ε is given by $\mathbb{C} \xrightarrow{e^{2\pi i\,\cdot}} \mathbb{C}^* \subseteq \mathcal{O}^*_X$. This shows that $L \in \mathrm{Pic}^0\,X$ can be represented by a factor of automorphy $f(\lambda, v)$ independent of $v \in V$.)

Exercise 3. Sections of line bundles. Let $X = V/\Lambda$ be a complex torus of dimension g that admits a *positive definite* hermitian form H such that $H(\Lambda, \Lambda) \subseteq \mathbb{Z}$. Let $E = \mathrm{Im}\,H$ be the corresponding alternating form. There exists a basis of Λ (called symplectic basis of Λ) such that E is given by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where $D = \mathrm{diag}(d_1, \ldots, d_g)$ with integers $d_i \geqslant 0$ and $d_i | d_{i+1}$. This induces a decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$. Let V_i the $\mathbb{R}$-linear span of Λ_i so $V = V_1 \oplus V_2$.

(a) Let be $\chi_0 : V \to \mathbb{C}$ such that $\chi_0(v) = e^{\pi i E(v_1, v_2)}$. Show that it is a semicharacter for H.

(b) Show that $\Lambda_2 \otimes \mathbb{C} = V$. Let $B : V \times V \to \mathbb{C}$ the $\mathbb{C}$-bilinear extension of the real symmetric form $H|_{V_2 \times V_2}$. Show that $(H - B)(v, w) = 0$ if $w \in V_2$ and $(H - B)(v, w) = 2iE(v, w)$ if $v \in V_2$.

If f is a factor of automorphy of a line bundle L, then the space of sections $H^0(X, L)$ of L can be identified naturally with the sections of the trivial bundle $\mathbb{C} \times V \to V$, that are invariant under the action of Λ, that is, the set of holomorphic functions $\vartheta : V \to \mathbb{C}$ such that

$$\vartheta(v + \lambda) = f(\lambda, v)\vartheta(v). \tag{1}$$

In the setting of the previous exercise, consider the line bundle L given by the Appell-Humbert data (H, χ_0).

(c) Assume that the following series is well-defined and absolutely convergent (this relies on the fact that $\mathrm{Re}(H - B)$ is positive definite on V_1)

$$\vartheta_H(v) = e^{\frac{\pi}{2}B(v,v)} \sum_{\mu \in \Lambda_1} e^{\pi(H-B)(v,\mu) - \frac{\pi}{2}(H-B)(\mu,\mu)}.$$

Show that ϑ_H satisfies (1) for the canonical factor of automorphy $f = a_{(H,\chi_0)}$.

Matrix presentations

Exercise 4. Riemann relations. Let $X = V/\Lambda$ be a complex torus of dimension g that admits a *positive definite* hermitian form H such that $H(\Lambda, \Lambda) \subseteq \mathbb{Z}$. Let $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ a symplectic basis of Λ for $E = \mathrm{Im}\, H$. i.e. with respect to this basis E is given by $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$ where $D = \mathrm{diag}\,(d_1, \ldots, d_g)$.

(a) Let $e_j = \frac{1}{d_j}\mu_j$, for $j = 1, \ldots, g$. Show that $\{e_j\}_j$ forms a basis of V (**Hint:** This is *(b)* of the previous Exercise). Denote $\Pi = (Z, D)$ and observe that $X = \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$.

(b) Show that ${}^tZ = Z$ and $\mathrm{Im}\, Z > 0$. (This are the Riemann relations with a symplectic basis).

(c) $(\mathrm{Im}\, Z)^{-1}$ is the matrix of H with respect to $e_1, \ldots, e_g$.

(d) Let $D = \mathrm{id}$, then in the setting of the previous Exercise, $\Lambda_1 = Z \times \mathbb{Z}^g$ and $\Lambda_2 = \mathbb{Z}^g$, and also $V_1 = Z\mathbb{R}^g$ and $V_2 = \mathbb{R}^g$. So $w \in W$, can be written as $w = Zw_1 + w_2$. Then the symmetric bilinear form B defined in *(b)* can be computed as $B(v, w) = {}^tv(\mathrm{Im}\, Z)^{-1}w$ and $(H - B)(v, w) = -2i{}^tvw_1$. So

$$\vartheta_H(v) = e^{\frac{\pi}{2}{}^tv(\mathrm{Im}\, Z)^{-1}v} \sum_{\eta \in \mathbb{Z}^g} e^{\pi i(2{}^tv\eta + {}^t\eta Z\eta)}.$$

(**Hint:** Replace η by $-\eta$.)

(e) Set

$$\overline{\vartheta}_Z(v) = \sum_{\eta \in \mathbb{Z}^g} e^{\pi i(2{}^tv\eta + {}^t\eta Z\eta)}.$$

Show that the zero locus of $\overline{\vartheta}_Z(v)$ is well-defined on X and it is called a *theta-divisor* on X.

Abelian varieties

Exercise 5. Algebraic constructions. Assume for simplicity that we work over an algebraically closed field k of characteristic 0.

(a) (See-saw principle) Let X and Y be varieties. Suppose X is complete. Let L and M be two line bundles on $X \times Y$. If for all closed points $y \in Y$ we have $L_y \cong M_y$ there exists a line bundle N on Y such that $L \cong M \otimes p^*N$, where $p : X \times Y \to Y$ is the projection onto Y. (**Hint:** By "semi-continuity", $h^0(X_y, L_y \otimes M_y) = 1$ for all close points, implies that $p_*(L \otimes M)$ is a line bundle).

It follows from the See-saw principle (via the Theorem of the cube) that if X is an abelian variety and Y a variety, then for every triple (f, g, h) of morphisms $Y \to X$ and for every line bundle L on X, we have

$$(f+g+h)^*L \cong (f+g)^*L \otimes (f+h)^*L \otimes (g+h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}. \tag{2}$$

Define $\operatorname{Pic}^0 X = \{M \in \operatorname{Pic} X \,|\, t_x^* M \cong M \text{ for all } x \in X\}$, where $t_x : X \to X$ is the translation map $t_x(y) = y + x$ (compare with the analytic description given by the Appell-Humbert Theorem).

(b) Let X be an abelian variety and $L \in \operatorname{Pic} X$. Deduce from (2) that

$$\varphi_L : X \to \operatorname{Pic}^0 X \quad \text{defined by } x \mapsto t_x^* L \otimes L^{-1}$$

is well-defined and is a homomorphism.

Define the Mumford line bundle $\mathscr{M}(L) := m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ on $X \times X$.
Define set theoretically $K(L) := \left\{ x \in X \,\middle|\, \mathscr{M}(L)|_{X \times \{x\}} \cong \mathcal{O}_X \right\}$.

(c) Show that $K(L) = \ker \varphi_L$ and deduce from the See-saw principle that $\mathscr{M}(L)|_{X \times K(L)} \cong \mathcal{O}_{X \times K(L)}$.

(d) Show that, if L is ample then $K(L)$ is a finite group.

(e) Let $M \in \operatorname{Pic}^0 X - \{\mathcal{O}_X\}$, show that $H^i(X, M) = 0$ for all i. (**Hint:** Show that $(-1)^*M = M^{-1}$, and use it to prove $H^0(X, M) = 0$. Then use Künneth formula to extend the result for $i > 0$).

Suppose that $L \in \operatorname{Pic} X$ is an ample line bundle. We have seen that φ_L is a homomorphism with finite kernel $K(L)$. Once, one prove that φ_L is surjective, we have seen that $\operatorname{Pic}^0 X$ is isomorphic to $X/K(L)$ as an abstract group. Section [13, §7] allows to give to $X/K(L) \cong \operatorname{Pic}^0 X$ an algebraic structure, such that, there exists a unique line bundle $\mathscr{P} \in \operatorname{Pic}(X \times \operatorname{Pic}^0 X)$ (the Poincaré line bundle), such that

$$\mathscr{M}(L) = (\operatorname{id} \times \varphi_L)^* \mathscr{P}, \tag{3}$$

i.e. $\mathscr{P}_{X \times \{M\}} \cong M \in \operatorname{Pic}^0 X$. The computation *(e)*, shows that the sheaves $R^i p_* \mathscr{P}$ are only supported at the origin. Using the full machinery of "semi-continuity" one can show that

$$R^i p_* \mathscr{P} = \begin{cases} k(0), & \text{if } i = g; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

(f) Let L be an ample line bundle. Show that $\chi(X \times X, \mathscr{M}(L)) = (-1)^g \chi(X, L)$.

(g) Use (3), (4), and the previous computation to show that $\deg \varphi_L = \chi(X, L)^2$.

(h) Use (4) to prove the following theorem [11, Thm. 2.2]: $\Phi_{\mathscr{P}} : \mathrm{D^b}(X) \to \mathrm{D^b}(\operatorname{Pic}^0(X))$, defined as $\Phi_{\mathscr{P}}(E) = \mathbf{R}p_{2*}(p_1^*E \otimes \mathscr{P})$, is an equivalence of derived categories. If we allow to interchange p_1 by p_2, we get more precisely that $\Phi_{\mathscr{P}} \circ \Phi_{\mathscr{P}} = (-1)^*[-g]$.

Exercise 6. Algebraic point of view of Exercise 1. An abelian variety is a group variety which, as a variety, is complete. Let X_1 and X_2 be abelian varieties and let $f : X_1 \to X_2$ be a morphism.

(b') The composition $h := t_{-f(0)} \circ f$ is a homomorphism of groups ($t_{x_0}(x) = x + x_0$ is the translation). (**Hint:** Use the Rigidity Lemma: Let X, Y and Z be algebraic varieties over a field k. Suppose that X is complete. If $f : X \times Y \to Z$ is a morphism with the property that, for some $y \in Y(k)$,

the fibre $X\times\{y\}$ is mapped to a point $z\in Z(k)$ then f factors through the projection $X\times Y\to Y$). As a corollary obtain that the group structure of an abelian variety is commutative.
Observe also that, we get $\mathrm{Hom}_{Sch/k}(X_1,X_2)=\mathrm{Hom}_{AV}(X,Y)\times Y(k)$.

(c') The following conditions are equivalent:

(i) f is surjective and $\dim X_1=\dim X_2$;

(ii) $\ker f$ is a finite group scheme and $\dim X_1=\dim X_2$;

(iii) f is a finite, flat and surjective morphism.

(**Hint:** You may use that quasi-finite morphism between two abelian varieties of the same dimension is flat. Also, if f is a morphism of finite type between two abelian varieties, then there is a non-empty open subset $U\subseteq Y$ such that either $f^{-1}(U)=\emptyset$ or the restricted morphism $f^{-1}(U)\to U$ is flat.)

(d') Let X be a g-dimensional abelian variety over a field k. Let ℓ be a prime number different from $\mathrm{char}(k)$. Then, the group scheme $X[\ell^n]:=\ker\ell^n$ has rank ℓ^{2ng}. We define the Tate-ℓ-module of X, to be the projective limit

$$T_\ell X:=\lim\left(0\xleftarrow{\ell}X[\ell]\xleftarrow{\ell}X[\ell^2]\xleftarrow{\ell}X[\ell^3]\xleftarrow{\ell}\cdots\right)$$

Then the $\mathbb{Z}_\ell$-linear map $T_\ell:\mathrm{Hom}(X_1,X_2)\otimes\mathbb{Z}_\ell\to\mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell X_1,T_\ell X_2)$ given by $f\otimes c\mapsto c\cdot T_\ell(f)$ is injective and has a torsion-free cokernel.

Moduli space

Exercise 7. The Siegel upper half space. Let $\mathbb{H}_g=\{Z\in M(g\times g,\mathbb{C})\,|\,Z={}^tZ,\ \mathrm{Im}\,Z>0\}$. Define $X_Z:=\mathbb{C}^g/(Z,D)\mathbb{Z}^{2g}$ and $H_Z:=(\mathrm{Im}\,Z)^{-1}$. The correspondence $Z\mapsto(X_Z,H_Z,\{\text{columns of }(Z,D)\})$ presents the Siegel upper half space $\mathbb{H}_G$ as a moduli space for polarized abelian varieties of (fixed) type D with symplectic basis.

(a) Show that there is an isomorphism $(X_Z,H_Z)\to(X_{Z'},H_{Z'})$ if, and only if, there is $M\in\mathrm{Sp}(D,\mathbb{Z})=\left\{M\in\mathrm{SL}(2g,\mathbb{Z})\,\middle|\,M\left(\begin{smallmatrix}0&D\\-D&0\end{smallmatrix}\right){}^tM=\left(\begin{smallmatrix}0&D\\-D&0\end{smallmatrix}\right)\right\}$ such that

$$Z'=(\alpha Z+\beta D)(\gamma Z+\delta D)^{-1}D,\qquad\text{where }M=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$$

We denote (**Hint:** Use the rational and the analytic representations of the isomorphism.)

(b) Show that the automorphism group of a polarized abelian variety (X,H) is finite.

(c) The map $\mathrm{Sp}(2g,\mathbb{R})$ acts on $\mathbb{H}_g$ as a group of biholomorphic automorphisms and the group homomorphism

$$\mathrm{Sp}(2g,\mathbb{R})\to\mathrm{Bihol}(\mathbb{H}_g)$$

has kernel $\{\pm1\}$.

Exercise 8. The "universal" family. Let $Z\in\mathbb{H}_g$ and consider the isomorphism of $\mathbb{R}$-vector spaces:

$$\begin{array}{rccc}j_Z: & \mathbb{R}^{2g} & \longrightarrow & \mathbb{C}^g\\ & x & \longmapsto & (Z,\mathrm{id})x\end{array}$$

and denote $\Lambda_D=\left(\begin{smallmatrix}\mathrm{id}&0\\0&D\end{smallmatrix}\right)\mathbb{Z}^{2g}$. Then Λ_D acts freely and properly discontinuously on $\mathbb{C}^g\times\mathbb{H}_g$ by

$$l(v,Z)=(v+j_Z(l),Z)\qquad\text{for }l\in\Lambda_D\text{ and }(v,Z)\in\mathbb{C}^g\times\mathbb{H}_g.$$

Consider $\mathbb{X}_D := (\mathbb{C}^g \times \mathbb{H}_g)/\Lambda_D \xrightarrow{p} \mathbb{H}_g$.

(a) Show that the fibre $p^{-1}(Z) = X_Z$ (Notation as in the previous exercise).

(b) Show that the map,

$$\begin{array}{rcl} \Lambda_D \times (\mathbb{C}^g \times \mathbb{H}_g) & \longrightarrow & \mathbb{C}^* \\ (l,(v,Z)) & \longmapsto & e^{-\pi i {}^t l^1 Z l^1 - 2\pi i {}^t v l^1}, \end{array}$$

where $l^1 \in \mathbb{R}^g$ denotes the vector of first g components of $l \in \mathbb{R}^{2g}$ is a factor of automorphy. Show that it defines a line bundle $\mathcal{L}$, such that $\mathcal{L}|_{X_Z} \cong L(H_Z, \chi_0)$.

(c) Recall that $\mathrm{Sp}(D,\mathbb{Z}) = \left\{ M \in \mathrm{SL}(2g,\mathbb{Z}) \,\middle|\, M \left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right) {}^t M = \left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right) \right\}$ and consider the action

$$\begin{array}{rrcl} M: & \mathbb{C}^g \times \mathbb{H}_g & \longrightarrow & \mathbb{C}^g \times \mathbb{H}_g \\ & (v,Z) & \longmapsto & ((\gamma Z + \delta)^{-1} v, M(Z)) \end{array} \qquad \text{where } M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right).$$

Show that it descends to an action on the family of abelian varieties $p : \mathbb{X}_D \to \mathbb{H}_g$.

(d) Show that we can consider the quotient $\bar{p} : \mathbb{X}_D / \mathrm{Sp}(D,\mathbb{Z}) \to \mathbb{H}_g / \mathrm{Sp}(D,\mathbb{Z})$, but the fiber $\bar{p}^{-1}(Z)$ over a fixed point $Z \in \mathbb{H}_g$ is the quotient of X_Z modulo the isotropy subgroup of $\mathrm{Sp}(D,\mathbb{Z})$ in Z.

We call $\mathcal{X}_D := \mathbb{X}_D / \mathrm{Sp}(D,\mathbb{Z})$ ($\mathcal{X}_g$ when $D = \mathrm{id}$). The full level structure

$$\Gamma(n) := \{ M \in \mathrm{Sp}(2,\mathbb{Z}) \,|\, M \equiv 1 \bmod n \} \subset \mathrm{Sp}(2g,\mathbb{Z})$$

acts freely on $\mathbb{H}_g$ for $n \geqslant 3$. Therefore, the previous construction replacing $\mathrm{Sp}(D,\mathbb{Z})$ by $\Gamma(n)$, allows us to construct a universal family $\mathcal{X}_g(n) \to \mathcal{A}_g(n) := \mathbb{H}_g/\Gamma(n)$. This shows that $\mathcal{A}_g(n)$ for $n \geqslant 3$ is a fine moduli space.

Exercise 9. Shioda modular surfaces. Recall $\Gamma(n) := \{ M \in \mathrm{Sp}(2,\mathbb{Z}) \,|\, M \equiv 1 \bmod n \}$. Consider

$$H(n) := \left\{ \left(\begin{smallmatrix} 1 & kn & ln \\ 0 & a & b \\ 0 & c & d \end{smallmatrix}\right); k,l \in \mathbb{Z}; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(n) \right\}$$

$H(n)$ acts on $\mathbb{C} \times \mathbb{H}_1$ by

$$\left(\begin{smallmatrix} 1 & kn & ln \\ 0 & a & b \\ 0 & c & d \end{smallmatrix}\right) : (t,z) \mapsto \left(\tfrac{t+knz+ln}{cz+d}, \tfrac{az+d}{cz+d} \right).$$

We have the following diagram:

$$\begin{array}{ccc} S^0(n) := (\mathbb{C} \times \mathbb{H}_1)/H(n), & & [t,z] \\ \downarrow & & \downarrow \\ X^0(n) := \mathbb{H}_1/\Gamma(n), & & [z]. \end{array}$$

(a) Show that the fibre over $z \in X^0(n)$ is $E_z = \mathbb{C}/(\mathbb{Z}z + \mathbb{Z})$.

(b) Show that the stabilizer at ∞ of $H(n)$ is

$$P = \left\{ \left(\begin{smallmatrix} 1 & kn & ln \\ 0 & 1 & rn \\ 0 & 0 & 1 \end{smallmatrix}\right); k,l,r \in \mathbb{Z} \right\} \cong \mathbb{Z}^3$$

(c) Consider P as the extension $1 \to P' \to P \to P'' \to 1$, where

$$P' = \left\{ \left(\begin{smallmatrix} 1 & 0 & ln \\ 0 & 1 & rn \\ 0 & 0 & 1 \end{smallmatrix}\right); l,r \in \mathbb{Z} \right\} \cong \mathbb{Z}^2 \qquad P'' = \left\{ \left(\begin{smallmatrix} 1 & kn & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right); k \in \mathbb{Z} \right\} \cong \mathbb{Z}.$$

Let W a neighbourhood of $\mathbb{C} \times \{\infty\}$. Show that

$$\begin{array}{rrcl} e: & W & \longrightarrow & (\mathbb{C}^*)^2 \\ & (t,z) & \longmapsto & (e^{\frac{2\pi i z}{n}}, e^{\frac{2\pi i t}{n}}) =: (u,w) \end{array}$$

is a partial quotient of W by the action of P', and P'' acts on $e(W) \subseteq (\mathbb{C}^*)^2$ by

$$\begin{pmatrix} 1 & kn & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : (x, y) \mapsto (x, x^{nk} y).$$

(d) Consider the lattice $M = \mathbb{Z}^2$ and the dual lattice $N = M^* = \mathbb{Z}^2$. In $M^* \otimes \mathbb{R}$ consider the following fan (i.e. collection Σ of strictly convex cones):

$$\begin{aligned} \sigma_k &= \mathbb{R}_{\geqslant 0}(k+1, 1) + \mathbb{R}_{\geqslant 0}(k, 1) \\ \rho_k &= \mathbb{R}_{\geqslant 0}(k, 1) \end{aligned}$$

If $\sigma \subset M^* \otimes \mathbb{R}$ is a cone, we call $\sigma^\vee = \{x \in M \otimes \mathbb{R} \mid \langle x, y \rangle \geqslant 0 \text{ for all } y \in \sigma\}$ the dual cone of σ. Show that

$$\begin{aligned} \sigma_k^\vee &= \mathbb{R}_{\geqslant 0}(1, -k) + \mathbb{R}_{\geqslant 0}(-1, k+1) \\ \rho_k^\vee &= \mathbb{R}_{\geqslant 0}(1, -k) + \mathbb{R}_{\geqslant 0}(-1, k+1) + \mathbb{R}_{\geqslant 0}(1, -k-1). \end{aligned}$$

Then

$$\begin{aligned} T_{\sigma_k} &:= \operatorname{Spec} \mathbb{C}[\sigma_k^\vee \cap M] \cong \mathbb{C}^2 \\ T_{\rho_k} &:= \operatorname{Spec} \mathbb{C}[\rho_k^\vee \cap M] \cong \mathbb{C} \times \mathbb{C}^* \\ T_{\{0\}} &= \operatorname{Spec} \mathbb{C}[M] \cong \mathbb{C}^*. \end{aligned}$$

(e) If $\rho \subset \tau$, by duality $\tau^\vee \subset \rho^\vee$, so we have induced maps $T_\rho \subset T_\tau$. Show that in our case

$$\begin{aligned} T_{\{0\}} &\hookrightarrow T_{\sigma_k}; \qquad (u, v) \mapsto (uv^{-k}, u^{-1} v^{k+1}) =: (u_k, v_k) \\ T_{\rho_k} &\hookrightarrow T_{\sigma_k}; \qquad (u_k, v_k) \mapsto (u_k, v_k) \\ T_{\rho_{k+1}} &\hookrightarrow T_{\sigma_k}; \qquad (u_{k+1}, v_{k+1}) \mapsto (v_k^{-1}, u_k v_k^2). \end{aligned}$$

(f) Define $\mathcal{T}_\Sigma$ as $\left(\coprod_{\tau \in \{0, \sigma_k, \rho_k\}} T_\tau \right) / \sim$, where two points in $x_1 \in T_{\varsigma_1}$ and $x_2 \in T_{\varsigma_2}$ are related if there exists a subcone $\varrho \subset \varsigma_1 \cap \varsigma_2$, such that $x_1 = x_2 \in T_\varrho$.
We have an embedding $T_{\{0\}} = (\mathbb{C}^*)^2 \hookrightarrow T_\Sigma$. Show that $T_\Sigma \setminus (\mathbb{C}^*)^2$ is a chain of $C_i \cong \mathbb{P}^1$ with $i \in \mathbb{Z}$, and such that $C_i \cap C_{i+1} = \{\text{pt}\}$. Show that the generator of P'' acts on the chain of $\mathbb{P}^1$ by sending C_i to C_{i+n}.

Then we can glue

$$S^0(n) \cup_{W/P} X_\Sigma / P'',$$

where $X_\Sigma = \overline{\mathring{e(W)}}$, and we have added a n-gon as a fiber over ∞, consisting of n curves $C_i \cong \mathbb{P}^1$.

(e) Show that in the quotient T_Σ / P'', we have $C_i^2 = -2$.

Since there is a transformation $g \in \mathrm{SL}(2, \mathbb{Z})$ which maps a cusp in $X(n)$ to ∞, we can repeat this procedure to obtain a compactification $S(n)$ of $S^0(n)$ fibred over $X(n)$ such that its singular fibres over each of the cusps of $X(n)$ are n-gons of smooth rational (-2)-curves.

(f) Consider the Hesse-pencil: $C_\lambda : x_0^3 + x_1^3 + x_2^3 - 3\lambda x_0 x_1 x_2 = 0$. Show that when C_λ is singular, then it is a triangle. this pencil has 9 fixed points. $S(3)$ is the blow-up of $\mathbb{P}^2$ on the 9 fixed points of the pencil. The induced map is the fibration $S(3) \to \mathbb{P}^1 \cong X(3)$.

Exercise 10. Semi-abelian varieites. A semi-abelian variety is an algebraic group G, which is the extension of an abelian variety A and a torus $T \cong (\mathbb{C}^*)^r$. Such group G is connected and commutative, and T is its unique maximal subtorus. The dimension of the torus T is called the rank of G, and $A = G/T$ its abelian part.

(a) Let $X := \mathrm{Hom}(T, \mathbb{C}) \cong \mathbb{Z}^r$ be the character group of the torus T. Show that the extensions $1 \to T \to G \to A \to 1$ are in $1-1$-correspondence with the homomorphisms

$$X \to \mathrm{Pic}^0 A.$$

(b) Let $Z \in \mathbb{H}_2$ and consider the lattice $\Lambda = \bigoplus_{i=1}^4 \mathbb{Z}e_i$, where e_i are the columns of the matrix (Z, id).

(c) Consider the abelian surface $A_Z = \mathbb{C}^2/\Lambda$. Show that $A_Z = (\mathbb{C}^*)^2/(\mathbb{Z}e_1 + \mathbb{Z}e_2)$, where $\mathbb{Z}e_1 + \mathbb{Z}e_2$ acts on $(\mathbb{C}^*)^2$ via

$$e_1 : (u, v) \mapsto (ue^{2\pi i z_{11}}, ve^{2\pi i z_{12}}) \qquad \text{and} \qquad e_2 : (u, v) \mapsto (ue^{2\pi i z_{12}}, ve^{2\pi i z_{22}}).$$

(d) Assume that $z_{22} \to \infty$. Then the action of e_2 is not free, but we can consider $G := \mathbb{C}^2/(\mathbb{Z}e_1 + \mathbb{Z}e_3 + \mathbb{Z}e_4)$. Show that

$$1 \to \mathbb{C}^* \to G \xrightarrow{\psi} E_{z_{11}} \to 1, \qquad \text{where } \psi([u, v]) = [u].$$

(e) Define $e := [z_{12}] \in E_{z_{11}}$. Show that the classifying morphism associated to the previous semi-abelian variety is

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathrm{Pic}^0 E_{z_{11}} \\ 1 & \longmapsto & \mathcal{O}_E(e - 0) \end{array}$$

(f) Show that $G \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(e-0))$ and we can consider the degenerate abelian surface as the quotient that identifies the points x of the zero section, with the points $x + e$ of the infinity section.

Exercise 11. Hodge line bundle and canonical bundle on $\mathcal{A}_g$. Assume that $g \geqslant 2$.

(a) Let $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{Sp}(2g, \mathbb{Z})$. Show that the holomorphic map $(M, Z) \mapsto f(M, Z) = \det(\gamma Z + \delta)^k$ satisfies the cocycle condition (it is a factor of automorphy),

$$f(M \cdot M', Z) = f(M, M'(Z)) f(M', Z) \qquad \text{for all } M, M' \in \mathrm{Sp}(2g, \mathbb{Z}) \text{ and } Z \in \mathbb{H}_g.$$

Since $\Gamma(3) = \{M \in \mathrm{SL}(2, \mathbb{Z}) \,|\, M \equiv \mathrm{id} \mod 3\} \subseteq \mathrm{Sp}(2g, \mathbb{Z})$ acts freely on $\mathbb{H}_g$, this factor of automorphy defines a line bundle on $\mathcal{A}_g(3) := \mathbb{H}_g/\Gamma(3)$, whose sections are holomorphic functions such that

$$F(M(Z)) = \det(\gamma Z + \delta)^k F(Z) \qquad \text{for all } M \in \Gamma(3) \text{ and } Z \in \mathbb{H}_g$$

((scalar) weight k modular forms for the full 3-level structure).

We can define the *Hodge vector bundle* as $\mathcal{E} := \pi_*(\Omega^1_{\mathcal{X}_g/\mathcal{A}_g})$ on $\mathcal{A}_g$ (in order to do that, we have to pretend that $\mathrm{Sp}(2g, \mathbb{Z})$ acts freely on $\mathbb{H}_g$, or use stacks -see Exercise 8). So, the fiber of the Hodge vector bundle over a point $[X] \in \mathcal{A}_g$ is the g-dimensional space of holomorphic 1-forms on X. We denote by $L := \det \mathcal{E}$ the corresponding determinant Hodge line bundle.

(b) We can lift the Hodge bundle through the quotient $p : \mathbb{H}_g \to \mathcal{A}_g$. Then, fiber of the Hodge bundle over Z is $p^*\mathcal{E}|_Z = H^0(X_Z, \Omega^1_{X_Z}) = \mathbb{C}dz_1 \oplus \ldots \oplus \mathbb{C}dz_g$ (i.e., $\mathcal{E}$ lifts to a trivial vector bundle on $\mathbb{H}_g$, but it is not trivial on the quotient $\mathcal{A}_g$). Let $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{Sp}(2g, \mathbb{Z})$. Show that the isomorphism

$$H^0(X_Z, \Omega^1_{X_Z}) \to H^0(X_{MZ}, \Omega^1_{X_{MZ}})$$

between complex g-vector spaces is given by the matrix $(\gamma Z + \delta)^{-1}$. This shows that the hodge line bundle L is the bundle of (scalar) modular forms of weight 1.

(c) To compute the canonical class of $\mathcal{A}_g$, consider the explicit volume form $\omega(Z) := \bigwedge_{i\leq j} Z_{ij}$ on $\mathbb{H}_g$. Show that $\omega(MZ) = \det(\gamma Z+\delta)^{-g-1}\omega(Z)$, which means that the canonical divisor is $K_{\mathcal{A}_g} = (g+1)L$.

Exercise 12. Picard group of $\mathcal{A}_g$. Consider $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties.

(a) Show that there are at most countably many proper analytic subvarieties A_i in the moduli space $\mathcal{A}_g$ such that $(X,\Theta)\in\mathcal{A}_g\setminus\bigcup_i A_i$ has endomorphism ring $\mathbb{Z}$. (**Hint:** consider for any $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in M(2g\times 2g,\mathbb{Z})$ the equation $Z(\gamma Z+\delta)=\alpha Z+\beta$ in $\mathbb{H}_g$.)

(b) Assume that the locus in $\mathcal{A}_g$ of principally polarized abelian varieties having endomorphism ring greater that $\mathbb{Z}$ has codimension 2. Then, prove that the smooth locus $\mathcal{A}_g^0$ of $\mathcal{A}_g$ can be characterized as locus of principally polarized abelian varieties (A,Θ) having automorphism group $\{\pm 1\}$ (i.e. $\mathrm{Aut}_{ppav}(A,H)=\{\pm1\}$). (**Hint:** use that $\mathcal{A}_{g,n}$ the moduli space of principally polarized abelian structure with level n-structure is smooth when $n\geqslant 3$, i.e $\Gamma_g(n) := \{\gamma\in\mathrm{Sp}(2g,\mathbb{Z})\,|\,\gamma\equiv \mathrm{id}_{2g}(mod\ n)\}$ acts freely on $\mathbb{H}_g$ if $n\geqslant 3$.)

(c) Assume that $g\geqslant 4$. Assuming that $H_1(\mathcal{A}_g^0,\mathbb{Z})=0$ and $H^2(\mathcal{A}_g^0,\mathbb{Z})\cong\mathbb{Z}$, show that $\mathrm{Pic}(\mathcal{A}_g^0)\cong\mathbb{Z}$. (**Hint**: Use that for $g\geqslant 4$, the boundary of the Satake compactification has codimension greater than 1.)

(d) Show that $H_1(\mathcal{A}_g^0,\mathbb{Z})=0$ and $H^2(\mathcal{A}_g^0,\mathbb{Z})\cong\mathbb{Z}$ (**Assume,** the following result of Borel: for any subgroup $\Gamma\subset\mathrm{Sp}(2g,\mathbb{Z})$ of finite index, we have $H^*(\Gamma,\mathbb{Q})=\mathbb{Q}[c_2,c_6,c_{10},\ldots]$ up to degree $\leqslant g-2$.)

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