Exercise 1. Homomorphisms of complex tori. Let $X_1 = V_1/\Lambda_1$ and $X_2 = V_2/\Lambda_2$ be two complex tori ($V_i$ are complex vector spaces and $\Lambda_i \subset V_i$ are lattices, i.e. discrete subgroups of maximal rank). Let $f : X_1 \to X_2$ be a holomorphic map.

(a) Show that there exists an affine map $\tilde{f} : V_1 \to V_2$, that induces $f$. (Hint: use Liouville Theorem).

(b) The composition $h := t_{-f(0)} \circ f$ is a homomorphism of groups ($t_{0,x_0}(x) = x + x_0$ is the translation).

The injective homomorphism $\text{Hom}(X_1, X_2) \to \text{Hom}_C(V_1, V_2)$ which sends $h \mapsto \tilde{h}$ is called the analytic representation of $\text{Hom}(X_1, X_2)$.

(c) If we consider $h : X_1 \to X_2$, then $h(X_1)$ is a subtorus of $X_2$ and the connected component $(\ker h)^0$ passing through 0 of $\ker h$ is a subtorus of $X_1$. $\ker h/(\ker h)^0$ is a finite group and $\dim X_1 = \dim(\ker h)^0 + \dim \text{im} h$.

(d) Show that $\text{Hom}(X_1, X_2) \cong \mathbb{Z}^m$ for some $m \leq 4 \dim X_1 \cdot \dim X_2$. (Hint: use the rational representation $\text{Hom}(X_1, X_2) \to \text{Hom}_\mathbb{Q}(\Lambda_1, \Lambda_2)$).

Line bundles on complex tori

Exercise 2. Factors of automorphy and line bundles on complex tori. Let $X = V/\Lambda$ be a complex torus. A holomorphic map $f : \Lambda \times V \to \mathbb{C}^*$ satisfying $f(\lambda + \mu, v) = f(\lambda, v) f(\mu, v)$ is called a factor of automorphy. Given an automorphy factor $f$ we can define the following action of $\Lambda$ on $V \oplus \mathbb{C}$,

$$\Lambda \ni \lambda : (v, t) \mapsto (v + \lambda, f(\lambda, v) \cdot t)$$

The quotient $L = (V \oplus \mathbb{C})/\Lambda$ is well-defined, and it is a line bundle over $X$. (This is a particular case of the isomorphism $H^1(\pi_1(X), H^0(\tilde{X}, O_{\tilde{X}}^*)) \cong \ker(\text{Hom}(\pi_1(X), O_X^*) \xrightarrow{\pi_*} H^1(X, O_X^*))$, where $\pi : \tilde{X} \to X$ is the universal cover of $X$.)

(a) Let $h : V \to \mathbb{C}^*$ be a holomorphic function. Show that $h(v + \lambda) h(v)^{-1}$ is a factor of automorphy that defines the trivial line bundle on $X$ (the cycles of this type are called coboundaries).

(b) Show that $H^2(X, \mathbb{Z}) \cong \text{Alt}^2(\Lambda, \mathbb{Z}) = \text{group of} \mathbb{Z}\text{-valued alternating 2-forms on} \Lambda$. (Hint: Use Künneth formula).

Consider the exponential exact sequence $0 \to \mathbb{Z} \to O_X \to O_X^* \to 0$. One one hand, it gives us the following exact sequence $0 \to \mathbb{Z} \to H^0(V, \pi^* O_X) \xrightarrow{\pi_*} H^0(V, \pi^* O_X^*) \to 0$. On the other hand, if we consider the coboundary maps of the cohomological long exact sequences, they are compatible through the following commutative diagram:

$$
\begin{array}{ccc}
H^1(\Lambda, H^0(V, \pi^* O_X^*)) & \xrightarrow{\delta} & H^2(\Lambda, \mathbb{Z}) \\
\cong & \cong & \\
\text{Pic} X \cong H^1(X, O_X^*) & \xrightarrow{\delta = c_1} & H^2(X, \mathbb{Z}) \cong \text{Alt}^2(\Lambda, \mathbb{Z})
\end{array}
$$

Therefore, given a factor of automorphy $f = e^{2\pi ig}$ defining $L \in \text{Pic} X$, the first Chern class $c_1(L)$ can be described in $\text{Alt}^2(\Lambda, \mathbb{Z})$ as

$$E_L(\lambda, \mu) = g(\mu, v + \lambda) + g(\lambda, v) - g(\lambda, v + \mu) - g(\mu, v) \quad \text{for all} \ v \in V.$$

(c) Show that $E_L$ is well-defined, i.e., the definition does not depend on $v \in V$. 

1
Recall that the image of $c_1$ is called the Neron-Severi group $\text{NS} X$ of $X$. A semicharacter for an hermitian form $H \in \text{NS} X$ is a map $\chi : \Lambda \to U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ such that

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{i\pi \text{Im} H(\lambda,\mu)},$$

so when $H = 0$ it is a character.

(g) Given $(H,\chi)$, where $\chi$ is a semicharacter (s.c.) for $H \in \text{NS} X$. Show that

$$a_{(H,\chi)}(\lambda, v) := \chi(\lambda)e^{\pi H(v,\lambda) + i\pi \text{Im} H(\lambda,\lambda)}$$

is a factor of automorphy (it is usually called canonical factor of $(H,\chi)$).

(h) Check that the following diagram commutes:

$$\begin{array}{ccc}
\{(H,\chi)\}_{\chi \text{ s.c. of } H \in \text{NS} X} & \xrightarrow{p_1} & \text{NS} X \\
\psi \downarrow & & \downarrow \\
\text{Pic} X & \xrightarrow{c_1} & \text{NS} X,
\end{array}$$

where $\psi(H,\chi) = a_{(H,\chi)}$.

(h) Finally to show the Appell-Humbert theorem, i.e. the following diagram:

$$\begin{array}{ccc}
0 & \xrightarrow{\phi} & \text{Hom}(\Lambda, U(1)) & \xrightarrow{p_1} & \{(H,\chi)\}_{\chi \text{ s.c. of } H \in \text{NS} X} & \xrightarrow{\psi} & \text{NS} X & \xrightarrow{c_1} & 0 \\
0 & \xrightarrow{} & \text{Pic}^0 X & \xrightarrow{} & \text{Pic} X & \xrightarrow{} & \text{Pic} X & \xrightarrow{c_1} & \text{NS} X & \xrightarrow{\psi} & 0,
\end{array}$$

we need to show that $\phi$ is an isomorphism (Hint: Consider $\text{Pic}^0 X \cong \text{Im}(H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X)) \cong \text{Im}(H^1(X,\mathbb{C}) \xrightarrow{\varepsilon} H^1(X,\mathcal{O}_X))$, where $\varepsilon$ is given by $\mathbb{C} \xrightarrow{\varepsilon} \mathbb{C}^{*} \subseteq \mathcal{O}_X$. This shows that $L \in \text{Pic}^0 X$ can be represented by a factor of automorphy $f(\lambda, v)$ independent of $v \in V$.)

**Exercise 3. Sections of line bundles.** Let $X = V/\Lambda$ be a complex torus of dimension $g$ that admits a positive definite hermitian form $H$ such that $H(\Lambda,\Lambda) \subseteq \mathbb{Z}$. Let $E = \text{Im} H$ be the corresponding alternating form. There exists a basis of $\Lambda$ (called symplectic basis of $\Lambda$) such that $E$ is given by the matrix

$$\begin{pmatrix}
0 & D \\
-D & 0
\end{pmatrix},$$

where $D = \text{diag}(d_1,\ldots,d_g)$ with integers $d_i \geq 0$ and $d_i | d_{i+1}$. This induces a decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$. Let $V_i$ the $\mathbb{R}$-linear span of $\Lambda_i$, so $V = V_1 \oplus V_2$.

(a) Let be $\chi_0 : V \to \mathbb{C}$ such that $\chi_0(v) = e^{\pi i E(v_1,v_2)}$. Show that it is a semicharacter for $H$. 

$$
\end{pmatrix},$$
Exercise 4. Riemann relations. Let $X = V/\Lambda$ be a complex torus of dimension $g$ that admits a positive definite hermitian form $H$ such that $H(\Lambda, \Lambda) \subseteq \mathbb{Z}$. Let $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ a symplectic basis of $\Lambda$ for $E = \text{Im } H$, i.e. with respect to this basis $E$ is given by \((0, D 0)\) where $D = \text{diag}(d_1, \ldots, d_g)$.

(a) Let $e_j = \frac{1}{2\pi} \mu_j$, for $j = 1, \ldots, g$. Show that $\{e_j\}_j$ forms a basis of $V$ (Hint: This is (b) of the previous Exercise). Denote $\Pi = (Z, D)$ and observe that $X = \mathbb{C}^g/\mathbb{Z}^g$.

(b) Show that $^tZ = Z$ and $\text{Im } Z > 0$. (This are the Riemann relations with a symplectic basis).

(c) $(\text{Im } Z)^{-1}$ is the matrix of $H$ with respect to $e_1, \ldots, e_g$.

(d) Let $D = \text{id}$, then in the setting of the previous Exercise, $\Lambda_1 = Z \times \mathbb{Z}^g$ and $\Lambda_2 = \mathbb{Z}^g$, and also $V_1 = Z \mathbb{R}^g$ and $V_2 = \mathbb{R}^g$. So $w \in W$, can be written as $w = Zw_1 + w_2$. Then the symmetric bilinear form $B$ defined in (b) can be computed as $B(v, w) = \nu(\text{Im } Z)^{-1} w$ and $(H - B)(v, w) = -2i \nu w_1$.

So
\[
\vartheta_H(v) = e^{\frac{2\pi i}{e} (\text{Im } Z)^{-1} v} \sum_{\eta \in \mathbb{Z}^g} e^{\pi i (2\nu \eta + \eta Z \eta)}.
\]

(Hint: Replace $\eta$ by $-\eta$.)

(e) Set
\[
\overline{\vartheta}_Z(v) = \sum_{\eta \in \mathbb{Z}^g} e^{\pi i (2\nu \eta + \eta Z \eta)}.
\]

Show that the zero locus of $\overline{\vartheta}_Z(v)$ is well-defined on $X$ and it is called a theta-divisor on $X$.

Abelian Varieties

Exercise 5. Algebraic constructions. Assume for simplicity that we work over an algebraically closed field $k$ of characteristic 0.
(a) (See-saw principle) Let $X$ and $Y$ be varieties. Suppose $X$ is complete. Let $L$ and $M$ be two line bundles on $X \times Y$. If for all closed points $y \in Y$ we have $L_y \cong M_y$ there exists a line bundle $N$ on $Y$ such that $L \cong N \otimes p^*N$, where $p : X \times Y \to Y$ is the projection onto $Y$. (Hint: By “semi-continuity”, $h^0(X_y, L_y \otimes M_y) = 1$ for all close points, implies that $p_*(L \otimes M)$ is a line bundle).

It follows from the See-saw principle (via the Theorem of the cube) that if $X$ is an abelian variety and $Y$ a variety, then for every triple $(f, g, h)$ of morphisms $Y \to X$ and for every line bundle $L$ on $X$, we have

$$ (f + g + h)^*L \cong (f + g)^*L \otimes (f + h)^*L \otimes (g + h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}. $$

Define $\text{Pic}^0X = \{ M \in \text{Pic}X \mid t_x^*M \cong M \text{ for all } x \in X \}$, where $t_x : X \to X$ is the translation map $t_x(y) = y + x$ (compare with the analytic description given by the Appell-Humbert Theorem).

(b) Let $X$ be an abelian variety and $L \in \text{Pic}X$. Deduce from (2) that

$$ \varphi_L : X \to \text{Pic}^0X \text{ defined by } x \mapsto t_x^*L \otimes L^{-1} $$

is well-defined and is a homomorphism.

Define the Mumford line bundle $\mathcal{M}(L) := m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ on $X \times X$.

Define set theoretically $K(L) := \{ x \in X \mid \mathcal{M}(L)|_{X \times \{x\}} \cong \mathcal{O}_X \}$.

(c) Show that $K(L) = \ker \varphi_L$ and deduce from the See-saw principle that $\mathcal{M}(L)|_{X \times K(L)} \cong \mathcal{O}_{X \times K(L)}$.

(d) Show that, if $L$ is ample then $K(L)$ is a finite group.

(e) Let $M \in \text{Pic}^0X - \{ \mathcal{O}_X \}$, show that $H^i(X, M) = 0$ for all $i$. (Hint: Show that $(-1)^iM = M^{-1}$, and use it to prove $H^0(X, M) = 0$. Then use K"unneth formula to extend the result for $i > 0$).

Suppose that $L \in \text{Pic}X$ is an ample line bundle. We have seen that $\varphi_L$ is a homomorphism with finite kernel $K(L)$. Once, one prove that $\varphi_L$ is surjective, we have seen that $\text{Pic}^0X$ is isomorphic to $X/K(L)$ as an abstract group. Section 13, §7 allows to give to $X/K(L) \cong \text{Pic}^0X$ an algebraic structure, such that, there exists a unique line bundle $\mathcal{P} \in \text{Pic}(X \times \text{Pic}^0X)$ (the Poincaré line bundle), such that

$$ (3) \quad \mathcal{M}(L) = (\text{id} \times \varphi_L)^*\mathcal{P}, $$

i.e. $\mathcal{P}_{X \times \{M\}} \cong M \in \text{Pic}^0X$. The computation (e), shows that the sheaves $R^i\varphi_*\mathcal{P}$ are only supported at the origin. Using the full machinery of “semi-continuity” one can show that

$$ (4) \quad R^i\varphi_*\mathcal{P} = \begin{cases} k(0), & \text{if } i = g; \\ 0, & \text{otherwise}. \end{cases} $$

(f) Let $L$ be an ample line bundle. Show that $\chi(X \times X, \mathcal{M}(L)) = (-1)^g\chi(X, L)$.

(g) Use (3), (4), and the previous computation to show that $\deg \varphi_L = \chi(X, L)^2$.

(h) Use (4) to prove the following theorem [11, Thm. 2.2]: $\Phi_\mathcal{P} : D^b(X) \to D^b(\text{Pic}^0(X))$, defined as $\Phi_\mathcal{P}(E) = R\varphi_*(p_1^*E \otimes \mathcal{P})$, is an equivalence of derived categories. If we allow to interchange $p_1$ by $p_2$, we get more precisely that $\Phi_\mathcal{P} \circ \Phi_\mathcal{P} = (-1)^g[-g]$.

Exercise 6. Algebraic point of view of Exercise 1. An abelian variety is a group variety which, as a variety, is complete. Let $X_1$ and $X_2$ be abelian varieties and let $f : X_1 \to X_2$ be a morphism.

(b') The composition $h := t_{-f(0)} \circ f$ is a homomorphism of groups ($t_{x_0}(x) = x + x_0$ is the translation).

(Hint: Use the Rigidity Lemma: Let $X$, $Y$ and $Z$ be algebraic varieties over a field $k$. Suppose that $X$ is complete. If $f : X \times Y \to Z$ is a morphism with the property that, for some $y \in Y(k)$,
the fibre $X \times \{ y \}$ is mapped to a point $z \in Z(k)$ then $f$ factors through the projection $X \times Y \to Y$.

As a corollary obtain that the group structure of an abelian variety is commutative.

Observe also that, we get $\text{Hom}_{\text{Sch}/k}(X_1, X_2) = \text{Hom}_{AV}(X, Y) \times Y(k)$.

(c') The following conditions are equivalent:

(i) $f$ is surjective and $\dim X_1 = \dim X_2$;
(ii) $\ker f$ is a finite group scheme and $\dim X_1 = \dim X_2$;
(iii) $f$ is a finite, flat and surjective morphism.

(Hint: You may use that quasi-finite morphism between two abelian varieties of the same dimension is flat. Also, if $f$ is a morphism of finite type between two abelian varieties, then there is a non-empty open subset $U \subseteq Y$ such that either $f^{-1}(U) = \emptyset$ or the restricted morphism $f^{-1}(U) \to U$ is flat.)

(d') Let $X$ be a $g$-dimensional abelian variety over a field $k$. Let $\ell$ be a prime number different from $\text{char}(k)$. Then, the group scheme $X[\ell^n]$ := $\ell^n$ has rank $\ell^{2ng}$. We define the Tate-$\ell$-module of $X$, to be the projective limit

$$T_\ell X := \lim_{\rightarrow} \begin{pmatrix} 0 & \ell & \ell & \ell & \cdots \end{pmatrix}$$

Then the $\mathbb{Z}_\ell$-linear map $T_\ell : \text{Hom}(X_1, X_2) \otimes \mathbb{Z}_\ell \to \text{Hom}_{\mathbb{Z}_\ell}(T_\ell X_1, T_\ell X_2)$ given by $f \otimes c \mapsto c \cdot T_\ell(f)$ is injective and has a torsion-free cokernel.

**Moduli space**

**Exercise 7. The Siegel upper half space.** Let $\mathbb{H}_g = \{ Z \in M(g \times g, \mathbb{C}) | Z = 'Z, \text{Im} Z > 0 \}$. Define $X_Z := \mathbb{C}^g/(Z, D)\mathbb{Z}^{2g}$ and $H_Z := (\text{Im} Z)^{-1}$. The correspondence $Z \mapsto (X_Z, H_Z, \{ \text{columns of (Z, D)} \})$ presents the Siegel upper half space $\mathbb{H}_G$ as a moduli space for polarized abelian varieties of (fixed) type $D$ with symplectic basis.

(a) Show that there is an isomorphism $(X_Z, H_Z) \to (X_{Z'}, H_{Z'})$ if, and only if, there is $M \in \text{Sp}(D, \mathbb{Z}) = \{ M \in \text{SL}(2g, \mathbb{Z}) | M \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}^t M = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \}$ such that

$$Z' = (\alpha Z + \beta D)(\gamma Z + \delta D)^{-1} D,$$

where $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

We denote (Hint: Use the rational and the analytic representations of the isomorphism.)

(b) Show that the automorphism group of a polarized abelian variety $(X, H)$ is finite.

(c) The map $\text{Sp}(2g, \mathbb{R})$ acts on $\mathbb{H}_g$ as a group of biholomorphic automorphisms and the group homomorphism

$$\text{Sp}(2g, \mathbb{R}) \to \text{Bihol}(\mathbb{H}_g)$$

has kernel $\{ \pm 1 \}$.

**Exercise 8. The “universal” family.** Let $Z \in \mathbb{H}_g$ and consider the isomorphism of $\mathbb{R}$-vector spaces:

$$j_Z : \mathbb{R}^{2g} \longrightarrow \mathbb{C}^g$$

$$x \longmapsto (Z, \text{id})x$$

and denote $A_D = \begin{pmatrix} \text{id} & 0 \\ 0 & 0 \end{pmatrix} \mathbb{Z}^{2g}$. Then $A_D$ acts freely and properly discontinuously on $\mathbb{C}^g \times \mathbb{H}_g$ by

$$l(v, Z) = (v + j_Z(l), Z) \quad \text{for } l \in A_D \text{ and } (v, Z) \in \mathbb{C}^g \times \mathbb{H}_g.$$
Consider $X_D := (C^g \times \mathbb{H}_g)/\Lambda_D \overset{p}{\rightarrow} \mathbb{H}_g$.

(a) Show that the fibre $p^{-1}(Z) = X_Z$ (Notation as in the previous exercise).

(b) Show that the map,

$$
\Lambda_D \times (C^g \times \mathbb{H}_g) \rightarrow C^*
$$

$$(l, (v, Z)) \mapsto e^{-\pi i l^1 Z^1 - 2\pi i l^1},$$

where $l^1 \in \mathbb{R}^g$ denotes the vector of first $g$ components of $l \in \mathbb{R}^{2g}$ is a factor of automorphy. Show that it defines a line bundle $\mathcal{L}$, such that $\mathcal{L}|_{X_Z} \cong L(H_Z, \chi_0)$.

(c) Recall that $\text{Sp}(D, \mathbb{Z}) = \{ M \in \text{SL}(2g, \mathbb{Z}) | M \left( \begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix} \right) \text{ mod } D, \chi = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \text{ } \text{ } k,l \in \mathbb{Z}; (a,b,c,d) \in \Gamma(n) \}$ and consider the action

$$
M : C^g \times \mathbb{H}_g \rightarrow C^g \times \mathbb{H}_g
$$

$$(v, Z) \mapsto ((\gamma Z + \delta)^{-1}v, M(Z))$$

where $M = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$.

Show that it descends to an action on the family of abelian varieties $p : X_D \rightarrow \mathbb{H}_g$. The fiber $\bar{p}^{-1}(Z)$ over a fixed point $Z \in \mathbb{H}_g$ is the quotient of $X_Z$ modulo the isotropy subgroup of $\text{Sp}(D, \mathbb{Z})$ in $Z$.

We call $X_D := X_D/\text{Sp}(D, \mathbb{Z})$ ($X_g$ when $D = \text{id}$). The full level structure

$$
\Gamma(n) := \{ M \in \text{Sp}(2, \mathbb{Z}) | M \equiv 1 \mod n \} \subset \text{Sp}(2g, \mathbb{Z})
$$

acts freely on $\mathbb{H}_g$ for $n \geq 3$. Therefore, the previous construction replacing $\text{Sp}(D, \mathbb{Z})$ by $\Gamma(n)$, allows us to construct a universal family $X_g(n) \rightarrow A_g(n) := \mathbb{H}_g/\Gamma(n)$. This shows that $A_g(n)$ for $n \geq 3$ is a fine moduli space.

Exercise 9. Shioda modular surfaces. Recall $\Gamma(n) := \{ M \in \text{Sp}(2, \mathbb{Z}) | M \equiv 1 \mod n \}$. Consider

$$
H(n) := \left\{ \left( \begin{smallmatrix} 1 & kn \\ 0 & c \end{smallmatrix} \right); k,l \in \mathbb{Z}; (a,b,c,d) \in \Gamma(n) \right\}
$$

$H(n)$ acts on $C \times \mathbb{H}_1$ by

$$(\frac{1}{0} \frac{kn}{c} \frac{ln}{d}) : (t, z) \mapsto \left( \frac{t+knz+ln}{c+zd}, \frac{az+zd}{c+zd} \right).$$

We have the following diagram:

$$S^0(n) := (C \times \mathbb{H}_1)/H(n), \quad [t, z]$$

$$\downarrow$$

$$X^0(n) := \mathbb{H}_1/\Gamma(n), \quad [z].$$

(a) Show that the fibre over $z \in X^0(n)$ is $E_z = \mathbb{C}/(\mathbb{Z}z + \mathbb{Z})$.

(b) Show that the stabilizer at $\infty$ of $H(n)$ is

$$P = \left\{ \left( \begin{smallmatrix} 1 & kn \\ 0 & c \end{smallmatrix} \right); k,l \in \mathbb{Z} \right\} \cong \mathbb{Z}^3$$

(c) Consider $P$ as the extension $1 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 1$, where

$$P' = \left\{ \left( \begin{smallmatrix} 1 & kn \\ 0 & c \end{smallmatrix} \right); k,l \in \mathbb{Z} \right\} \cong \mathbb{Z}^2 \quad P'' = \left\{ \left( \begin{smallmatrix} 1 & kn \\ 0 & c \end{smallmatrix} \right); k \in \mathbb{Z} \right\} \cong \mathbb{Z}.$$

Let $W$ a neighbourhood of $C \times \{\infty\}$. Show that

$$e : W \rightarrow (C^*)^2$$

$$(t, z) \mapsto (e^{2\pi i t}, e^{2\pi i z}) =: (u, w)$$
is a partial quotient of $W$ by the action of $P'$, and $P''$ acts on $e(W) \subseteq (C^*)^2$ by
\[
\begin{pmatrix} 1 & kn & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : (x, y) \mapsto (x, x^{kn}y).
\]

(d) Consider the lattice $M = \mathbb{Z}^2$ and the dual lattice $N = M^* = \mathbb{Z}^2$. In $M^* \otimes \mathbb{R}$ consider the following fan (i.e. collection $\Sigma$ of strictly convex cones):
\[
\begin{align*}
\sigma_k &= \mathbb{R}_{\geq 0} (k + 1, 1) + \mathbb{R}_{\geq 0} (k, 1) \\
\rho_k &= \mathbb{R}_{\geq 0} (k, 1)
\end{align*}
\]
If $\sigma \subset M^* \otimes \mathbb{R}$ is a cone, we call $\sigma^\vee = \{ x \in M \otimes \mathbb{R} | \langle x, y \rangle \geq 0 \text{ for all } y \in \sigma \}$ the dual cone of $\sigma$. Show that
\[
\begin{align*}
\sigma_k^\vee &= \mathbb{R}_{\geq 0} (1, -k) + \mathbb{R}_{\geq 0} (-1, k + 1) \\
\rho_k^\vee &= \mathbb{R}_{\geq 0} (1, -k) + \mathbb{R}_{\geq 0} (-1, k + 1) + \mathbb{R}_{\geq 0} (1, -k - 1).
\end{align*}
\]
Then
\[
\begin{align*}
T_{\sigma_k} &=: \text{Spec } \mathbb{C}[\sigma_k^\vee \cap M] \cong \mathbb{C}^2 \\
T_{\rho_k} &=: \text{Spec } \mathbb{C}[\rho_k^\vee \cap M] \cong \mathbb{C} \times \mathbb{C}^* \\
T_{(0)} &=: \text{Spec } \mathbb{C}[M] \cong \mathbb{C}^*.
\end{align*}
\]
(e) If $\rho \subset \tau$, by duality $\tau^\vee \subset \rho^\vee$, so we have induced maps $T_\rho \subset T_\tau$. Show that in our case
\[
\begin{align*}
T_{(0)} &\hookrightarrow T_{\sigma_k}; \quad (u, v) \mapsto (uv^{-k}, u^{-1}v^{k+1}) =: (u_k, v_k) \\
T_{\rho_k} &\hookrightarrow T_{\sigma_k}; \quad (u_k, v_k) \mapsto (u_k, v_k) \\
T_{\rho_{k+1}} &\hookrightarrow T_{\sigma_k}; \quad (u_{k+1}, v_{k+1}) \mapsto (v_k^{-1}, u_k v_k^2).
\end{align*}
\]
(f) Define $T_{\Sigma}$ as $\left( \prod_{\tau \in \{0, \sigma_k, \rho_k\}} T_{\tau} \right) / \sim$, where two points in $x_1 \in T_{\sigma_1}$ and $x_2 \in T_{\sigma_2}$ are related if there exists a subcone $\rho \subset \sigma_1 \cap \sigma_2$, such that $x_1 = x_2 \in T_{\sigma}$. We have an embedding $T_{(0)} = (\mathbb{C}^*)^2 \hookrightarrow T_{\Sigma}$. Show that $T_{\Sigma} \setminus (\mathbb{C}^*)^2$ is a chain of $C_i \cong \mathbb{P}^1$ with $i \in \mathbb{Z}$, and such that $C_i \cap C_{i+1} = \{ pt \}$. Show that the generator of $P''$ acts on the chain of $\mathbb{P}^1$ by sending $C_i$ to $C_{i+n}$.
Then we can glue
\[
S^0(n) \cup_{W/P} X_{\Sigma}/P'';
\]
where $X_{\Sigma} = \overline{e(W)}$, and we have added a $n$-gon as a fiber over $\infty$, consisting of $n$ curves $C_i \cong \mathbb{P}^1$.

(e) Show that in the quotient $T_\Sigma/P''$, we have $C_1^2 = -2$.
Since there is a transformation $g \in \text{SL}(2, \mathbb{Z})$ which maps a cusp in $X(n)$ to $\infty$, we can repeat this procedure to obtain a compactification $\bar{S}(n)$ of $S^0(n)$ fibred over $X(n)$ such that its singular fibres over each of the cusps of $X(n)$ are $n$-gons of smooth rational $(-2)$-curves.

(f) Consider the Hesse-pencil: $C_\lambda : x_0^3 + x_1^3 + x_2^3 - 3\lambda x_0 x_1 x_2 = 0$. Show that when $C_\lambda$ is singular, then it is a triangle. this pencil has 9 fixed points. $S(3)$ is the blow-up of $\mathbb{P}^2$ on the 9 fixed points of the pencil. The induced map is the fibration $S(3) \rightarrow \mathbb{P}^1 \cong X(3)$.

Exercise 10. Semi-abelian varieties. A semi-abelian variety is an algebraic group $G$, which is the extension of an abelian variety $A$ and a torus $T \cong (\mathbb{C}^*)^r$. Such group $G$ is connected and commutative, and $T$ is its unique maximal subtorus. The dimension of the torus $T$ is called the rank of $G$, and $A = G/T$ its abelian part.
(a) Let \( X := \text{Hom}(T, \mathbb{C}) \cong \mathbb{Z}^* \) be the character group of the torus \( T \). Show that the extensions \( 1 \to T \to G \to A \to 1 \) are in \( 1-1 \)-correspondence with the homomorphisms \( X \to \text{Pic}^0 A \).

(b) Let \( Z \in \mathbb{H}_2 \) and consider the lattice \( \Lambda = \bigoplus_{i=1}^{2} \mathbb{Z}e_i \), where \( e_i \) are the columns of the matrix \((Z, \text{id})\).

(c) Consider the abelian surface \( A_Z = \mathbb{C}^2/\Lambda \). Show that \( A_Z = (\mathbb{C}^*)^2/(\mathbb{Z}e_1 + \mathbb{Z}e_2) \), where \( \mathbb{Z}e_1 + \mathbb{Z}e_2 \) acts on \((\mathbb{C}^*)^2\) via
\[
e_1 : (u, v) \mapsto (ue^{2\pi i z_1}, ve^{2\pi i z_2}) \quad \text{and} \quad e_2 : (u, v) \mapsto (ue^{2\pi i z_2}, ve^{2\pi i z_2}).
\]

(d) Assume that \( z_{22} \to \infty \). Then the action of \( e_2 \) is not free, but we can consider \( G := \mathbb{C}^2/(\mathbb{Z}e_1 + \mathbb{Z}e_3 + \mathbb{Z}e_4) \). Show that
\[
1 \to \mathbb{C}^* \to G \xrightarrow{\psi} E_{z_{22}} \to 1, \quad \text{where} \ \psi([u, v]) = [u].
\]

(e) Define \( e := [z_{12}] \in E_{z_{22}} \). Show that the classifying morphism associated to the previous semi-abelian variety is
\[
\mathbb{Z} \to \text{Pic}^0 E_{z_{22}}, \quad 1 \mapsto \mathcal{O}_E(e - 0)
\]

(f) Show that \( G \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(e - 0)) \) and we can consider the degenerate abelian surface as the quotient that identifies the points \( x \) of the zero section, with the points \( x + e \) of the infinity section.

Exercise 11. Hodge line bundle and canonical bundle on \( A_g \). Assume that \( g \geq 2 \).

(a) Let \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}) \). Show that the holomorphic map \((M, Z) \mapsto f(M, Z) = \det(\gamma Z + \delta)^k\) satisfies the cocycle condition (it is a factor of automorphy),
\[
f(M \cdot M', Z) = f(M, M'(Z))f(M', Z) \quad \text{for all} \ M, M' \in \text{Sp}(2g, \mathbb{Z}) \text{ and } Z \in \mathbb{H}_g.
\]

Since \( \Gamma(3) = \{ M \in \text{SL}(2, \mathbb{Z}) \mid M \equiv \text{id} \mod 3 \} \subseteq \text{Sp}(2g, \mathbb{Z}) \) acts freely on \( \mathbb{H}_g \), this factor of automorphy defines a line bundle on \( A_g(3) := \mathbb{H}_g/\Gamma(3) \), whose sections are holomorphic functions such that
\[
F(M(Z)) = \det(\gamma Z + \delta)^k F(Z) \quad \text{for all } M \in \Gamma(3) \text{ and } Z \in \mathbb{H}_g
\]
((scalar) weight \( k \) modular forms for the full 3-level structure).

We can define the Hodge vector bundle as \( \mathcal{E} := \pi_* (\Omega^1_{X_Z/A_g}) \) on \( A_g \) (in order to do that, we have to pretend that \( \text{Sp}(2g, \mathbb{Z}) \) acts freely on \( \mathbb{H}_g \), or use stacks -see Exercise 8). So, the fiber of the Hodge vector bundle over a point \([X] \in A_g \) is the \( g \)-dimensional space of holomorphic 1-forms on \( X \). We denote by \( L := \det \mathcal{E} \) the corresponding determinant Hodge line bundle.

(b) We can lift the Hodge bundle through the quotient \( p : \mathbb{H}_g \to A_g \). Then, fiber of the Hodge bundle over \( Z \) is \( p^* \mathcal{E}|_Z = H^0(X_Z, \Omega^1_{X_Z}) = \mathbb{C}dz_1 \oplus \ldots \oplus \mathbb{C}dz_g \) (i.e., \( \mathcal{E} \) lifts to a trivial vector bundle on \( \mathbb{H}_g \), but it is not trivial on the quotient \( A_g \)). Let \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}) \). Show that the isomorphism
\[
H^0(X_Z, \Omega^1_{X_Z}) \to H^0(X_{MZ}, \Omega^1_{X_{MZ}})
\]
between complex \( g \)-vector spaces is given by the matrix \( (\gamma Z + \delta)^{-1} \). This shows that the hodge line bundle \( L \) is the bundle of (scalar) modular forms of weight 1.
(c) To compute the canonical class of $A_g$, consider the explicit volume form $\omega(Z) := \bigwedge_{i<j} Z_{ij}$ on $\mathbb{H}_g$. Show that $\omega(MZ) = \det(\gamma Z + \delta)^{-g-1}\omega(Z)$, which means that the canonical divisor is $K_{A_g} = (g+1)L$.

Exercise 12. Picard group of $A_g$. Consider $A_g$ the moduli space of principally polarized abelian varieties.

(a) Show that there are at most countably many proper analytic subvarieties $A_i$ in the moduli space $A_g$ such that $(X, \Theta) \in A_g \setminus \bigcup A_i$ has endomorphism ring $\mathbb{Z}$. (Hint: consider for any $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in M(2g \times 2g, \mathbb{Z})$ the equation $Z(\gamma Z + \delta) = \alpha Z + \beta$ in $\mathbb{H}_g$.)

(b) Assume that the locus in $A_g$ of principally polarized abelian varieties having endomorphism ring greater that $\mathbb{Z}$ has codimension 2. Then, prove that the smooth locus $A_g^0$ of $A_g$ can be characterized as locus of principally polarized abelian varieties $(A, \Theta)$ having automorphism group $\{\pm 1\}$ (i.e. $\text{Aut}_{ppav}(A, H) = \{\pm 1\}$). (Hint: use that $A_{g,n}$ the moduli space of principally polarized abelian structure with level $n$-structure is smooth when $n \geq 3$, i.e $\Gamma_g(n) := \{1 \in \text{Sp}(2g, \mathbb{Z}) | \gamma \equiv \text{id}_g (\text{mod} \ n)\}$ acts freely on $\mathbb{H}_g$ if $n \geq 3$.)

(c) Assume that $g \geq 4$. Assuming that $H_1(A_g^0, \mathbb{Z}) = 0$ and $H^2(A_g^0, \mathbb{Z}) \cong \mathbb{Z}$, show that $\text{Pic}(A_g^0) \cong \mathbb{Z}$. (Hint: Use that for $g \geq 4$, the boundary of the Satake compactification has codimension greater than 1.)

(d) Show that $H_1(A_g^0, \mathbb{Z}) = 0$ and $H^2(A_g^0, \mathbb{Z}) \cong \mathbb{Z}$ (Assume, the following result of Borel: for any subgroup $\Gamma \subset \text{Sp}(2g, \mathbb{Z})$ of finite index, we have $H^*(\Gamma, \mathbb{Q}) = \mathbb{Q}[c_2, c_6, c_{10}, \ldots]$ up to degree $\leq g-2$.)

References