Oral Quals Notes

Howard Nuer

April 29, 2012

1 Intersection Theory

1.1 Rational Equivalence

1.1.1 Orders of Zeros and Poles

We define the order function associated to a codimension one subvariety V of a vareity X to be the homomorphism $ord_V : K(X)^* \to \mathbb{Z}$ defined by $ord_V(r) = l_A(A/(r))$ for $r \in A$, where $K(X)$ is the function field of X, $A = \mathcal{O}_{V,X}$, and l_A means length as an A-module.

1.1.2 Cycles and Rational Equivalence

Let X be a scheme. We define a k-cycle on X to be a finite formal sum $\sum n_i[V_i]$, where V_i are k-dimensional subvarieties of X and the n_i are integers. We write Z_kX for the free abelian group generated by all of the k-dimensional subvarieties of X. We define for any $(k+1)$-dimensional subvariety W and any $r \in K(X)^*$ the k-cycle on X

$$[div(r)] = \sum ord_V(r)[V]$$

, where the sum is over all codimension one subvarieties V of W. A k-cycle α is rationally equivalent to zero ($\alpha \sim 0$) if there are a finite number of $(k+1)$-dimensional subvarieties W_i of X and $r_i \in K(W_i)^*$ such that $\alpha = \sum[div(r_i)]$. Since $[div(r^{-1})] = -[div(r)]$, the cycles rationally equivalent to zero form a subgroup Rat_kX. We define the k-th Chow group $A_k(X) := Z_k(X)/Rat_k(X)$. We denote by $Z_*(X)$ and $A_*(X)$ the direct sum over all dimensions of the individual Chow groups.

A positive cycle is one which is not zero and whose coefficients are all positive. A cycle class is positive if it can be represented by a positive cycle. We have the following easy but useful proposition:

Proposition 1.1. *For any scheme, we have $A_k(X) \cong A_k(X_{red})$. If X is a disjoint union of schemes $X_1, ..., X_l$, then both Z_* and A_* decompose as direct sums in the obvious way. If X_1 and X_2 are closed subschemes of X, then there are exact sequences*

$$A_k(X_1 \cap X_2) \to A_k(X_1) \oplus A_k(X_2) \to A_k(X_1 \cup X_2) \to 0.$$

1.1.3 Push-forward of Cycles

Let $f : X \to Y$ be a proper morphism. Then for any subvariety V the image $f(V)$ is then a closed subvariety of Y. Since the restricted morphism is surjective (and thus dominant), we have an induced inclusion of function fields $K(f(V)) \subset K(V)$, which is finite if V and $f(V)$ have the same dimension. We define $f_*[V] = [K(V) : K(f(V))][f(V)]$ if $\dim V = \dim f(V)$ and 0 otherwise. This extends linearly to a homomorphism $f_* : Z_k(X) \to Z_k(Y)$. In characteristic 0, if $\dim f(V) = \dim V$, then V is generically a covering of $f(V)$ with $[K(V) : K(f(V))]$ sheets by generic flatness. In this case pushforward agrees with the topological one.

The big theorem here, which isn't so surprising but important nonehteless, is that this homomorphism passes to rational equivalence. This is the:

Theorem 1.2. *If $f : X \to Y$ is a proper morphism, and α is a k-cycle rationally equivalent to zero, then $f_*\alpha$ is rationally equivalent to zero.*

Proof. Clearly we may just consider the case $\alpha = [div(r)]$, where r is a rational function on a subvariety of X. We may replace X by this subvariety and Y by the image $f(X)$, so we may assume Y is a variety and that f is surjective. The theorem then follows from the following technical lemma. □

Lemma 1.3. *If $f : X \to Y$ is a proper, surjective morphism of varietiesand $r \in K(X)^*$. Then*

(a) $f_*[div(r)] = 0$ if $\dim(Y) < \dim(X)$;

(b) $f_*[div(r)] = [div(N(r))]$ if $\dim(Y) = \dim(X)$, where $N(r)$ is the norm of the finite extension.

The proof of this lemma is somehwat lengthy and involves numerous cases so we don't provide it here. It is not difficult though.

Definition 1.4. If X is a complete scheme over K, and $\alpha = \sum_P n_P[P]$ is a zero-cycle, then we define the **degree** of α to be

$$\deg \alpha = \int_X \alpha = \sum_P n_P[\kappa(P) : K].$$

Equivalently, $\deg \alpha = p_*(\alpha)$, where $p : X \to \text{Spec } K$ is the structure morphism. From this description it follows from the theorem that the degree of a cycle is well-defined under rational equivalence. It follows from functoriality that if $f : X \to Y$ is a proper morphism between complete schemes, then $\int_X \alpha = \int_Y f_*(\alpha)$. We extend this definition (using the second description) to all cycles.

There are two interesting examples that follow from the theory so far. The first is that Bezout's theorem for plane curves follows quite easily.

Theorem 1.5. *If F and G are projective plane curves of degrees m, n with no common components, then $\sum i(P, F \cdot G) = mn$*

Proof. Suppose F is an irreducible plane curve. If two polynomials G and G' have the same degree n, then G/G' defines a rational function r on the curve F and thus $0 = \deg([div(G/G')]) = \sum i(P, F \cdot G) - \sum i(P, F \cdot G')$. Thus taking $G' = L^n$ for linear L, we can reduce to the case of intersecting with a line. Likewise we can reduce to the case that F is a line as well. □

The second example is that the theorem fails if separatedness is not included in the definition of proper. Of course the push-forward would still be defined, we really only need the morphism to be closed, but the following example demonstrates the failure of the theorem: Suppose we take X to be the projective line with a double point. Then we know this is not-separated but the structure morphism $f : X \to \text{Spec } K$ is closed. Take $r = x_1/x_0$. Then $[div(r)] = (1 : 0)_1 + (1 : 0)_2 - (0 : 1)$ so that the pushforward is $f_*([div(r)]) = 1 \cdot pt$, which is not zero in $A_0(\text{Spec } K)$.

1.1.4 Cycles of Subschemes

For any scheme X with irreducible components X_i we define the fundamental cycle of X to be $[X] = \sum m_i[X_i]$, where $m_i = l_{\mathcal{O}_{X_i,X}}(\mathcal{O}_{X_i,X})$. If X is purely k-dimensional then $Z_k(X) = A_k(X)$ is the free abelian group generated by the cycles of these components.

1.1.5 Alternate Definition of Rational Equivalence

For any scheme X, we can define rational equivalence in an equivalent way. Let $X \times \mathbb{P}^1$ be the product and $V \subset X \times \mathbb{P}^1$ a $(k+1)$-dimensional subvariety which projects dominantly onto the second factor. Denote this morphism by f. Then for any point P rational over the base field, $f^{-1}(P)$ is isomorphic to a subvariety $X \times \{P\}$ and projects isomorphicaly by the first projection p onto a subvariety of X. We denote this subvariety by $V(P)$, and it's clear that $p_*([f^{-1}(P)]) = [V(P)]$. It is also clear that

$$[f^{-1}(0)] - [f^{-1}(\infty)] = [div(f)].$$

Thus

$$[V(0)] - [V(\infty)] = p_*[div(f)],$$

which is rationally equivalent to zero from the Theorem. The following proposition shows that this definition is equivalent to the first one.

Proposition 1.6. *A cycle α in $Z_k(X)$ is rationally equivalent to zero iff there are $(k+1)$-dimensional subvarieties $V_1, ..., V_t$ of $X \times \mathbb{P}^1$, such that the projection from V_i to $\mathbb{P}^1$ dominant, with*

$$\alpha = \sum [V_i(0)] - [V_i(\infty)]$$

in $Z)k(X)$.

Proof. One direction is clear from the comment preceding the proposition. For the other direction, it clearly suffices to prove it for the divisor of one rational function. So suppose $\alpha = [div(r)], r \in K(W)^*, W$ a $(k+1)$-dimensional subvariety of X. Then r defines a rational map from W to $\mathbb{P}^1$. If U is the open subset of W over which this map is actually a morphism, then let V be the closure inside $X \times \mathbb{P}^1$ of its graph. The first projection p maps V birationally and properly onto W. Let f be the induced morphism from V to $\mathbb{P}^1$. Then $[V(0)] - [V(\infty)] = p_*[div(f)] = [div(r)]$ by Lemma 1.3 and the fact that p is of degree 1. The proposition is thus proved. □

Using the proposition we can give a slightly more geometric definition of rationally equivalent cycles. First we say that a cycle $Z = \sum n_i[V_i]$ on $X \times \mathbb{P}^1$ projects dominantly to $\mathbb{P}^1$ if each variety V_i which actually appears in this sum does. We see $Z(0) = \sum n_i[V_i(0)], Z(\infty) = \sum n_i[V_i(\infty)]$. We can show that any two cycles α, α' are rationally equivalent iff there is a positive $(k+1)$-cycle Z on $X \times \mathbb{P}^1$ projecting dominantly to $\mathbb{P}^1$, and a positive k-cycle β on X such that $Z(0) = \alpha + \beta$ and $Z(\infty) = \alpha' + \beta$. Clearly if this condition is satisfied then the two cycles differ by a cycle as in the Proposition. Conversely, if $\alpha - \alpha'$ is of the form in the proposition, then it's equal to $Z'(0) - Z'(\infty)$ for some positive Z'. Choose a positive cycle β so that $\gamma := \alpha - Z'(0) + \beta$ is positive. Write $\gamma = \sum[V_i]$ and define $Z = Z' + \sum[V_i \times \mathbb{P}^1]$. Then $Z(0) = Z'(0) + \sum[V_i] = Z'(0) + \gamma = \alpha + \beta$ and $Z(\infty) = Z'(\infty) + \gamma = Z'(0) + \alpha' - \alpha + \gamma = \alpha + \beta - \alpha + \alpha' = \alpha' + \beta$. Thus the alternate definition essentialy says that these two cycles are components of a sort of "linear system" plus a fixed component. So rational equivalence is a special case of algebraic equivalence (which we'll see later) when the paramter space is a rational curve.

There are other definitions which use the symmetric power or Chow varieties. In addition one can use this definition to describe the kernel of the degree map. All of these are related to the above example and use the following useful fact (also used in Igusa's lemma in Hartshorne):

Lemma 1.7. *Any two points on a variety X can be connected by the image of a smooth curve.*

Proof. We may assume that X is projective by Chow's lemma. Then take the blow-up $\pi : X' \to X$ at the two points x, y. This is projective and by Bertini's we may take a generic hyperplane section to get an irreducible codimension 1 subvariety of X'. As long as $\dim X \geq 2$ (which is required for the blow-up to make sense and for the statement of the lemma to be nontrviial) then $\pi^{-1}(x), \pi^{-1}(y)$ are both of dimension at least 1 and thus intersect any hyperplane. Repeating this intersection process we arrive at an irreducible curve on X' (in fact Bertini's guarantees this is a smooth curve) whose image in X is an irreducible curve connecting x and y. It's normalization is the desired smooth curve. □

1.1.6 Flat Pull-back

For the sake of Intersection Theory we assume all flat morphisms have a relative dimension. Examples of flat morphisms with relative dimensions are open immersions, morphisms from a bundle of some sort to its base, morphisms from $Y \times Z$ to Y where Z is of pure dimension k, and dominant morphisms from a variety to a smooth curve.

Definition 1.8. We define the flat pull-back of a flat morphism $f : X \to Y$ of relative dimension n by $f^*([V]) = [f^{-1}(V)]$, where V is a k-dimensional subvariety of Y. By basic results on flatness $f^{-1}(V)$ is a purely $(k+n)$-dimensional subscheme. We extend by linearity to get a morphism $f^* : Z_k(Y) \to Z_{k+n}(X)$.

We record the following algebraic lemma which is useful in dealing with flatness:

Lemma 1.9. *Let $f : A \to B$ be a flat local homomorphism of local rings. Then the induced morphism from Spec B to Spec A is surjective. If A and B are zero-dimensional, then*

$$l_B(B) = l_A(A) l_B(B/mB),$$

where m is the maximal ideal of A.

Proof. We must show that for any prime ideal P there is a prime ideal of B whose restriction to A is P. Since by stability of flatness under base change B/PB is flat over A/P, we may assume $P = 0$ and that A is a domain. Then all non-zero elements are non-zero-divisors, and by flatness remain that way on B. These form a multiplicative subset $U \subset B$ and thus a prime not meeting U and maximal with respect to this property is a prime that restricts to zero. This proves the first part. For the second, take a chain

$$A = I_0 \supset I_1 \supset ... \supset I_r = 0,$$

of ideals of A whose quotients as modules are $I_{i-1}/I_i \cong A/m$ (it must a quotient of A by a maximal ideal and there's only one). Then extending coefficients by B, we get

$$B = BI_0 \supset BI_1 \supset ... \supset BI_r = 0$$

with quotients $BI_{i-1}/BI_i \cong I_{i-1}/I_i \otimes_A B \cong B/mB$, by flatness. Taking length over B we get $l_B(B) = r \cdot l_B(B/mB)$ from the additivity of length. □

We can use this to see that our definition of flat pull-back would also work for any subscheme:

Lemma 1.10. *If $f : X \to Y$ is a flat morphism, and Z is a closed subscheme of Y, then $f^*[Z] = [f^{-1}(Z)]$.*

Proof. Let W be an irreducible component of $f^{-1}(Z)$. Then let $V = \overline{f(W)}$. This is an irreducible component of Z as we show. If it weren't, then it would be contained in an irreducible component V' of Z. By the lemma applied to $B = \mathcal{O}_{W,f^{-1}(Z)}$, $A = \mathcal{O}_{V,Z}$, we get that there must be an irreducible subvariety W' of $f^1(Z)$ which contains W and dominates V'. But since W is an irreducible component of $f^{-1}(Z)$ we see that it must equal W' and thus $V' = V$. So we see that the irreducible components of $f^{-1}(Z)$ are mapped to the irreducible components of Z. We just need to check that the multiplicities agree. Writing out the definition we see that $f^*[Z] = \sum l_{\mathcal{O}_{V_i,Z}}(\mathcal{O}_{V_i,Z})[f^{-1}(V_i)]$. Now for each i, $[f^{-1}(V_i)] = \sum_j l_{W_{ij},f^{-1}(V_i)}[W_{ij}]$. These lengths are just the factors $l_B(B/mB)$ from the lemma. Thus we get the desired equality of multiplicities since their product is precisely $l_{W_{ij},f^{-1}(Z)}$. □

Besides showing that the definition extends as is, this also demonstrates functoriality directly.

One might wonder how flat pull-back and proper push-forward are related. The following proposition shows they in fact commute in the ideal situation.

Proposition 1.11. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a fibre square with g flat and f proper. Then g' is flat and f' is proper, and $f'_*g'^*\alpha = g^*f_*\alpha$, for all $\alpha \in Z_*(X)$.

Proof. The first assertion follows from stability of flatness and properness under base change. From some changing of base we may assume $\alpha = [X], X, Y$ are varieties, and f is surjective. Suppose $f_*[X] = d[Y]$. Then $g'^*[X] = [X']$ and $g^*[Y] = [Y']$, so we need to show that $f'_*[X'] = d[Y']$. This is a local question, which uses Lemma 1.12 below. To get there notice that if Y'_j is an irreducible component of Y' then since g is flat it must dominate Y by the proof of the previous lemma. Moreover $f'_*[X'] = \sum l_{X'_i,X'} f'_*[X'_i]$. For the equality we may restrict ourselves to those components which dominate a component of Y'. For each component Y'_j, let A be its local ring, which is Artinian. Then since X' is the fiber product, we get that the preimage of Y'_j (or really it's generic point) is given by $B = A \otimes_K L$, where $K = K(Y)$ and $L = K(X)$. Now let $r = l_A(A)$. Then since L is a d-dimensional vector space over K, we get that $B = A^d$ as an A-module. By the additivity of length we get that its length as an A-module is rd. Thus B has finite length over an Artinian ring and is thus Artinian. Since it's a ring we may decompose it as a direct sum of local Artinian rings, $B = \oplus B_i$. Then $rd = \sum l_A(B_i)$. Now by Lemma 1.12, we get that $l_A(B_i) = l_{B_i}(B_i)d_i$, where d_i is the degree of the residue field extension. But now we're pretty much done since we have, restricting only to those components which dominate Y'_j, $\sum l_{X'_{ij},X'} d_{ij}[Y'_j] = rd[Y'_j]$. Summing over the components of Y' we get the result. □

Lemma 1.12. *If $A \to B$ is a local homomorphism of local rings. Let d be the degree of the residue field extension. A non-zero B-module M has finite length over A iff $d < \infty$ and M has finite length over B in which case $l_A(M) = d \cdot l_B(M)$.*

Proof. By additivity we may reduce to the case $M = B/q$ where q is the maximal ideal of B. Then if p is the maximal ideal of A we have that $l_A(M) = l_{A/p}(B/q)$ since the morphism is a local morphism. But this is just d since length agrees with dimension over a field. □

As before, the big theorem here is that f^* passes to rational equivalence.

Theorem 1.13. *Let $f : X \to Y$ be a flat morphism of relative dimension n and α a k-cycle on Y rationally equivalent to zero, then so is $f^*\alpha$. Thus f^* passes to a homomorphism on Chow groups.*

Proof. By the alternate definition of rational equivalence, we may assume $\alpha = [V(0)] - [V(\infty)]$, for V a subvariety of $Y \times \mathbb{P}^1$ mapping dominantly (and thus flatly) to $\mathbb{P}^1$ by the induced morphism g. Let $W = (f \times 1)^{-1}(V) \subset X \times \mathbb{P}^1$, and let h be the restriction of the projection to $\mathbb{P}^1$. Furthermore, let $p : X \times \mathbb{P}^1 \to X$ and $q : Y \times \mathbb{P}^1 \to Y$ be the first projections. Then $f^*\alpha = f^*q_*([g^{-1}(0)] - [g^{-1}(\infty)]) = p_*(f \times 1)^*([g^{-1}(0)] - [g^{-1}(\infty)])$ by Lemma 1.11. By Lemma 1.10 we see this is just $p_*([h^{-1}(0)] - [h^{-1}(\infty)]$. Now let $W_1, ..., W_t$ be the irreducible components of W and h_i the restriction of h to h_i. Write $[W] = \sum m_i[W_i]$. Then $[div(h_i)] = [h_i^{-1}(0)] - [h_i^{-1}(\infty)]$ and p_* preserves rational equivalence, so we just need to show that $[h^{-1}(P)] = \sum m_i[h_i^{-1}(P)]$, for $P = 0$ or ∞. This is just the following algebraic lemma which we don't show. □

Lemma 1.14. *Let X be a purely n-dimensional scheme with irreducible components X_i and multiplicities m_i, and D be an effective Cartier divisor on X. Then $[D] = \sum m_i[D \cap X_i]$.*

This theorem fails if we don't assume flat morphisms have constant relative dimensions and this lemma fails if X isn't pure-dimensional. For example if $X \subset \mathbb{A}^3$ is defined by (zx, zy) and E is defined by $z - x$.

Proposition 1.15. *If $f : X' \to X$ is finite and flat, then f^*f_* is multiplication by d, the degree of the morphism.*

Proof. For any affine open $U \subset X$, $f^{-1}(U)$ is affine corresponding to a flat finitely generated $\mathcal{O}_X(U)$-module. Thus it's locally free and we may shrink U to make it a finitely generated free $\mathcal{O}_X(U)$-module. The rank of this module is d. Clearly it suffices to show the claim on subvarieties V of X. Then $f_*f^*[V] = f_*[f^{-1}(V)]$. Now f restricted to $f^{-1}(V) \to V$ is still flat and finite by stability under base-change. Thus all irreducible components of $f^{-1}(V)$ dominate V, so we just need to show that the sum of their lengths in $f^{-1}(V)$ is d. This is very similar to the proof of Proposition 1.11. It also directly follows from the fact that we can take an open affine around the generic point of V and get a direct sum of rank d. □

1.1.7 An Exact Sequence

Proposition 1.16. *Let Y be a closed subscheme of a scheme X and let U be the complement. Let i, j be the respective inclusions. Then the sequence $A_k(Y) \to A_k(X) \to A_k(U) \to 0$ is exact.*

Proof. The exactness on cycles is clear from the definition of these maps as inclusions and from the fact that taking a closed subvariety of U and taking it's closure gives a closed subvariety of X whose intersection with U is the original variety. Clearly the induced maps on Chow groups are exact at the rightmost spot. So we just need to show exactness in the middle. Suppose $\alpha \in Z_k(X)$ has $j^*\alpha = 0$ in $A_k(U)$. Then $j^*\alpha = \sum[div(r_i)]$ for $r_i \in K(W_i)^*$ where W_i are subvarieties of U. Taking the closure inside X actually doesn't change the function field, so we have $j^*(\alpha - \sum[div(r_i)]) = 0$ and thus $\alpha = \sum[div(r_i)] + i_*(\beta)$ for some $\beta \in Z_k(Y)$. Passing to Chow groups we're done. □

1.1.8 Affine Bundles

The following proposition is the first result in an interesting story about the intersection theory of bundles.

Proposition 1.17. *Let $p : E \to X$ be an affine bundle of rank n. Then the flat pull-back $p^* : A_k(X) \to A_{k+n}(E)$ is surjective.*

Proof. Choose a trivialization of E over an open affine U with complement Y. Then we have a commutative diagram,

$$\begin{array}{ccccccc}
A_*(Y) & \longrightarrow & A_*(X) & \longrightarrow & A_*(U) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
A_*(p^{-1}Y) & \longrightarrow & A_*(E) & \longrightarrow & A_*(p^{-1}U) & \longrightarrow & 0
\end{array}$$

where the rows are exact from the previous section and the columns are flat pull-backs by base-change. A diagram chase shows that it suffices to show the result for Y and U. Noetherian induction on Y would show it for Y. So we just need to prove it for U, and thus we may assume that X is affine and $E = X \times \mathbb{A}^n$. Since p factors through $X \times \mathbb{A}^n$, we may in fact assume $n = 1$. Clearly it suffices to show surjectivity on subvarieties V of E of dimension $k+1$. Furthermore, by Proposition 1.11, we can assume $X = \overline{p(V)}$ is an affine variety (closed subschemes of affine schemes are affine) onto which V is mapped dominantly. So let A be the coordinate ring of X and $K = K(X)$ its field of fractions. Let Q be the prime ideal in $A[t]$ corresponding to V. If $\dim X = k$, then $V = E$ so $p^*[X] = [E] = [V]$. So assume $\dim X = k+1$. Since $V \neq E$ and it dominates X it is a non-zero ideal and thus remains so in $K[t]$. Let r generate this ideal in the PID $K[t]$. Now r is a polynomial with coefficients of the form a/b with $a, b \in A$. If we let A' denote the localization of A at all the a's and b's appearing in representations of the coefficients, then we get that $QA'[t] = (r'(t))$, where now $r'(t)$ is a monic polynomial in one variable with coefficients in A'. Thus over the open subset Spec $A' \subset X$, we get that V and $[div(r'(t))]$ are equal. Since $[div(r(t))] = [div(r'(t))] - \sum n_i V_i$, where the sum comes from components of the vanishing of the various coefficients we localized by, we have that $[V] - [div(r(t))] = [V] - [div(r'(t))] + \sum n_i [V_i]$. Note that since the V_i come from the vanishing of elements of A, they are in fact of the form $W_i \times \mathbb{A}^1$, where $W_i = p(V_i)$. Thus $\sum n_i V_i = p^*[\sum n_i W_i]$, so $[V]$ is rationally equivalent to an element of $p^* A_k(X)$ which is what we wanted to show. □

We immediately get from this proposition the following somewhat obvious fact:

Corollary 1.18. $A_k(\mathbb{A}^n) = 0$ *for* $k < n$ *and* $A_n(\mathbb{A}^n) \cong \mathbb{Z}$.

The analogue for Projective space also follows although it's slightly more interesting.

Corollary 1.19. *Denote by* L^k *a* k*-dimensional linear subspace of* $\mathbb{P}^n$*. Then* $A_k(\mathbb{P}^n) = \mathbb{Z}[L^k] \cong \mathbb{Z}$.

Proof. The fact that $[L^k]$ generates the k-th Chow group follows by induction on codimension and the proposition. Indeed for codim 1, we have an exact sequence

$$A_{n-1}(L^{n-1}) \to A_{n-1}(\mathbb{P}^n) \to A_{n-1}(\mathbb{P}^n - L^{n-1}) \to 0,$$

but $\mathbb{P}^n - L^{n-1} \cong \mathbb{A}^n$, so this last term is zero. The induction step involves finding a dimenson $k+1$ linear subspace containing L^k. Then everything follows as above. To see that $[L^k]$ is not torsion, we first note that for $k = n-1$ this follows from the divisors section of Hartshorne. For $k < n-1$ suppose that $d[L^k] = \sum n_i [div(r_i)]$ with $r_i \in K(V_i)$. Let Z be the union of an L^{k+1} containing L^k and the V_i. From a dimension count, we can find an $(n-k-2)$-dimensional linear subspace disjoint from Z. Let f be the projection from Z to $\mathbb{P}^{k+1}$. Then L^k is sent a L^k in the target. We see from the hyperplane case and the fact that pushforward preserves rational equivalence that we must have $d = 0$. So we're done. □

1.1.9 Exterior Products

We can define a homomorphism $\times : Z_k(X) \otimes Z_l(Y) \to Z_{k+l}(X \times Y)$ by $[V] \times [W] = [V \times W]$, where the latter may not be irreducible if the ground field isn't algebraically closed. The following proposition allows us descend to the Chow groups:

Proposition 1.20. *(a) If either* $\alpha \sim 0$ *or* $\beta \sim 0$*, then* $\alpha \times \beta \sim 0$.

(b) Let $f : X' \to X, g : Y' \to Y$ *be morphisms,* $f \times g$ *the induced morphisms on the products. Then (i) for* f, g *proper,* $(f \times g)_*(\alpha \times \beta) = f_*\alpha \times g_*\beta$*, and (ii) for* f, g *flat* $(f \times g)^*\alpha \times \beta = f^*\alpha \times g^*\beta$.

Proof. For (b) we may reduce to the case where one of the morphisms is the identity by factoring $f \times g = (f \times id_Y) \circ (id_{X'} \times g)$. This is then obvious for pullbacks since we don't need to actually decomponse into irreducible components.

For push-forwards it isn't difficult either. For part (a), we may assume by symmetry that $\alpha \sim 0$. Clearly it suffices to consider the case when $\beta = [W]$ for W a subvariety of Y. If I could show that $\alpha \times [W] \sim 0$ in $Z_*(X \times W)$ then by part (b) I'd be done since rational equivalence is preserved by proper push-forward. So we may assume $Y = W$. But then $\alpha \times [W] = p^*(\alpha)$, where $p : X \times W \to X$ is the flat projection. Now it just follows from the fact that flat pull-back preserves rational equivalence. □

1.2 Intersecting with Divisors

Let D be a Cartier divisor, then we can associate to it a Weil divisor $[D] = \sum ord_V(D)[V]$. Note that the Wiel divisors up to linear equivalence are just $A_{n-1}(X)$. Here $ord_V(D)$ is defined by $ord_V(f_\alpha)$ where f_α is a local equation for D on U_α with $U_\alpha \cap Y \neq \varnothing$. This is well-defined from the definition of Cartier divisor. Moreover, from the material in Hartshorne, we know this passes to linear equivalence. Thus we get a homomorphism $CaCl(X) \to A_{n-1}(X)$. For integral schemes, we may associate $CaCl(X)$ with $Pic(X)$, the group of isomorphism classes of line bundles. Now we present three examples to show that in general this homomorphism is neither injective nor surjective.

Example 1.21. *From the results of Hartshorne, if X is normal (resp. locally factorial) then $CaDiv(X) \to Z_{n-1}(X)$ and $CaCl(X) \to A_{n-1}(X)$ are injective (resp. isomorphisms).*

Example 1.22. *Let X be defined by $y^2z = x^3$ in $\mathbb{P}^2$. Then by (6.11.4) in Hartshorne, we see that $A_0(X) \cong \mathbb{Z}$ and the homomorphism $Pic(X) \to A_0(X)$ is surjective with kernel $\mathbb{G}_a$. If X is given instead by $y^2z = x^3 + x^2z$ then the kernel is $\mathbb{G}_m$*

Example 1.23. *Let X be the cone in $\mathbb{A}^3$ defined by $z^2 = xy$. Then the line $V : x = z = 0$ obviously defines a Weil divisor, but as in (6.11.3) of Hartshorne we see that it is not a Cartier divisor, so the homomorphism isn't surjective. In fact in this case we have $Pic(X) = 0$ and $A_1(X) \cong \mathbb{Z}/2$.*

1.2.1 Lines Bundles and Pseudo-divisors

Because of difficulties pulling back Cartier divisors under arbitrary morphisms of schemes, Fulton defines a generalization which still carries enough information to be of use and doesn't have these deficiencies.

Definition 1.24. A **pseudo-divisor** on a scheme X is a triple (L, Z, s), where L is a line bundle, Z is a closed subset, and s is a section of L over $X - Z$ which is nowhere vanishing. We say that another triple (L', Z', s') defines the same pseudo-divisor if $Z = Z'$ and there is an isomorphism $\sigma : L \to L'$ which takes s to s' over $X - Z$. A Cartier divisor determines a pseudo-divisor $(\mathcal{O}_X(D), |D|, s_D)$, where s_D is the section of $\mathcal{O}_X(D)$ determined by the local equations f_i determining D. We that a Cartier divisor D *represents* a pseudo-divisor (L, Z, s) if $|D| \subset Z$ and there is an isomorphism from $\mathcal{O}_X(D) \to L$ which off Z takes s_D to s.

One might wonder how restrictive being represented by a Cartier divisor is. It turns out it's not at all restrcitive as we have the:

Lemma 1.25. *If X is a variety, any pseudo-divisor (L, Z, S) is represented by some Cartier divisor. If $Z \neq X$ the Cartier divisor is uniquely determined, and if $Z = X$ it is determined up to linear equivalence.*

Proof. For existence, let $\{U_\alpha\}$ be an open affine covering of X for which L has transition functions $g_{\alpha\beta}$. Pick an open affine U_{α_0} and set $f_\alpha = g_{\alpha_0\alpha}$. Then $f_\alpha/f_\beta = g_{\alpha\beta}$, so $\{(U_\alpha, f_\alpha)\}$ determines a Cartier divisor D with $\mathcal{O}_X(D) \cong L$. If $Z = X$ we're done since there is no restriction on the sections. If $Z \neq X$ we have to adjust our D a little bit. Set $U = X - Z$. Then s is a section of $L \cong \mathcal{O}_X(D)$ which is nowhere vanishing on U. Since $\mathcal{O}_X(D)$ is really a subsheaf of the constant sheaf of rational functions, we get that s is given by nowhere vanishing regular functions s_α on $U \cap U_\alpha$ such that $s_\alpha = g_{\alpha\beta}s_\beta$. Then s_α/f_α is well defined and is equal to s_β/f_β on $U \cap U_\alpha \cap U_\beta$. So this defines a rational function $r \in K(X)$ and we have $r = s_\alpha/f_\alpha$ for all α. Let $D' = D + div(r)$. Then D' is given by local equations $f'_\alpha = rf_\alpha = s_\alpha$. Moreover, this defines the same line bundle as D. Thus $(\mathcal{O}_X(D'), |D'|, s_{D'})$ represents our pseudo-divisor.

For uniqueness, if D and D' are two Cartier divisors with local equations f_α and f'_α, respectively, which represent the same pseudo-divisor, then they must have isomorphic line bundles and thus are linear equivalent. This is indeed all we can say if $Z = X$ since the sections don't determine anything. From linear equivalence we get that they differ by a principal divisor, that is there is an $r \in K(X)^*$ such that $f_\alpha = rf'_\alpha$. If $U = X - Z \neq \varnothing$, then since $s_D = s_{D'}$ on U, we have that $f'_\alpha = f_\alpha$ on $U \cap U_\alpha$ for every α, i.e. $r = 1$ on U. But then $r = 1 \in K(X)^*$. Thus $D = D'$ as Cartier divisors. □

Definition 1.26. If D is a pseudo-divisor on an n-dimensional variety X, then we associate a Weil divisor $[D] \in A_{n-1}(|D|)$ as follows: we first take a Cartier divisor which represents D and let $[D]$ be its Weil divisor. If $|D| \neq X$, then $[D]$ is well-defined as the Cartier divisor is unique. If $|D| = X$, then all the Cartier divisors representing it are linearly equivalent and thus their associated Weil divisors are rationaly equivalent.

We define addition as follows: for $D = (L, Z, s), D' = (L', Z', s')$, we define $D + D' = (L \otimes L', Z \cup Z', s \otimes s')$. Inverses are defined by $-D = (L^{-1}, Z, 1/s)$. Pull-backs are defined by $f^*D = (f^*L, f^{-1}(Z), f^*s)$. This is all functorial and agrees with previous definitions when appropriate.

1.2.2 Intersecting with Pseudo-divisors

Definition 1.27. We define the intersection class $D \cdot V \in A_{k-1}(|D| \cap V)$ of D and a k-dimensional subvariety V as follows: let $j : V \to X$ be the inclusion, then $D \cdot [V] := [j^*D]$. We extend this linearly to any cycle α, in which case $D \cdot \alpha$ is a cycle in $A_{k-1}(|\alpha| \cap |D|)$. Note that if $\mathcal{O}_X(D)$ restricts to $|D|$ as a trivial line bundle, then intersecting with D defines a homomorphism on the level of cycles. Note that for any V with $V \subset |D|$, $D \cdot [V] = 0$.

We now show that this intersection product satisfies the expected properties.

Proposition 1.28. *(a)* $D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha' \in A_{k-1}(|D| \cap (|\alpha| \cup |\alpha'|))$.

(b) $(D + D')\alpha = D \cdot \alpha + D' \cdot \alpha \in A_{k-1}((|D| \cup |D'|) \cap |\alpha|)$.

(c) Let $f : X' \to X$ *be a proper morphism,* α *a* k*-cycle on* X'*, and* g *the induced morphism* $f^{-1}(|D|) \cap |\alpha| \to |D| \cap f(|\alpha|)$*. Then* $g_*(f^*D \cdot \alpha) = D \cdot f_*(\alpha) \in A_{k-1}(|D| \cap f(|\alpha|))$.

(d) Let $f : X' \to X$ *be a flat morphism of relative dimension* n *and* g *the induced morphism* $f^{-1}(|D| \cap |\alpha|) \to |D| \cap |\alpha|$*. Then* $f^*D \cdot f^*\alpha = g^*(D \cdot \alpha) \in A_{k+n-1}(f^{-1}(|D| \cap |\alpha|)$.

(e) If $\mathcal{O}_X(D)$ *is trivial then* $D \cdot \alpha = 0 \in A_{k-1}(|\alpha|)$.

Proof. (a) follows from the way we extended the definition linearly. (b) follows from the fact that pull-backs are additive (mutliplicative) and taking Weil divisors is a homomorphism.

For (c) we can reduce to the case $\alpha = [V]$, $V = X'$, $X = f(V)$ by functoriality. Obviously we may also restrict to the case of a Cartier divisor as this is how the intersection product is defined. In this case, f^*D is also Cartier since $D \subset X$ and not the other way around. Then g is really just the morphism f. Moreover, $f^*D \cdot [X'] = [f^*D]$. The right hand side is $D \cdot \deg(X'/X)[X] = \deg(X'/X)[D]$, unless it's zero for dimensional reasons, in which the left hand side is also 0 for the same reason. So what we need to show is $f_*[f^*D] = \deg(X'/X)[D]$. Since the Weil divisor associated to a Cartier divisor is determined locally from its local equations, we may look over each open set U_α, where $D|_{U_\alpha} = div(f_\alpha)$. Then $f^*D|_{f^{-1}(U_\alpha)} = div(f^*f_\alpha)$. But then $f|_{f^{-1}(U_\alpha)}$ is still proper by base-change stability of properness, and by Lemma 1.3, $f_*[div(f^*f_\alpha)] = [div(N(f^*f_\alpha))] = [div(f_\alpha^d)] = d[div(f_\alpha)]$, where $d = \deg(X'/X)$. Patching everything together gives (c).

We can reduce (d) to the case of $\alpha = [V]$ a subvariety, $V = X$, and D a Cartier divisor by applying the projection formula and the commutativity of pull-back and push-forward to the commutative fibre square

$$\begin{array}{ccc} f^{-1}(V) & \longrightarrow & X' \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}.$$

So now the left side reads $f^*D \cdot f^*[X] = f^*D \cdot [X'] = [f^*D]$, while the ride side is $f^*(D \cdot [X]) = f^*[D]$. Again since Weil divisors are determined by the local data defining the Cartier divisors, we may assume D is the difference of two effective divisors, and since both sides are additive we may assume D is effective. But then this just follows from Lemma 1.10.

Finally, to show (e) we may by linearity assume $\alpha = [V]$. But then from the definitions we're taking the Weil divisor corresponding to the trivial line bundle, which is obviously rationally equivalent to 0. □

A useful fact to be used when intersection with pull-backs of divisors to products of schemes is the following:

Corollary 1.29. *Let $p : X \times Y \to X$ be the first projection. Then $p^*D \cdot \alpha \times \beta = (D \cdot \alpha) \times \beta$.*

Proof. We first reduce to the case $\beta = [Y]$ with Y a variety. Suppose $\beta = [V]$ with V a subvariety. Consider the inclusion $i : X \times V \to X \times Y$. Then if we can show the claim in this case we're done, since $p^*D \cdot (\alpha \times [V]) = p^*D \cdot i_*(\alpha \times [V]) = i_*(i^*p^*D \cdot \alpha \times [V]) = i_*((D \cdot \alpha) \times [V]) = D \cdot \alpha \times [V]$. But in this case it's clear since $p^*D \cdot \alpha \times [Y] = p^*D \cdot p^*\alpha = p^*(D \cdot \alpha) = D \cdot \alpha \times [Y]$ by (d) of the proposition. □

1.2.3 Commutativity of Intersection Classes

We would hope that regardless of the order in which we intersect with pseudo-divisors it will still give the same class of cycles. Indeed this is the case.

Theorem 1.30. *Let $D \cdot [D'] = D' \cdot [D]$.*

Proof. We have to separate the proof into cases. The first is very algebraic at the heart of it, while the rest involves some beautiful geometric techniques.

Case I: Suppose D and D' are effective and intersect properly, i.e. there are no codimension 1 subvarieties contained in the intersection $|D| \cap |D'|$. Let W be any codimension two subvariety of X and A its local ring. Let a, a' be the local equations for D and D' in A, respectively. Ultimately we want to find the coefficient of $[W]$ inside $D \cdot [D']$. We first must consider the codimension 1 subvarieties V containing W, which are represented by height 1 primes $P \subset A$. Now the coefficient of $[V]$ in $[D']$ is given by $l_{A_P}(A_P/a'A_P)$. The coefficient of $[W]$ in $D \cdot [V]$ is given by $l_{A/P}(A/(P + aA))$, so the coefficient of $[W]$ in $D \cdot [D']$ is $\sum_P l_{A_P}(A_P/a'A_P) l_{A/P}(A/(P + aA))$. Doing the same for the right side boils things down to pure algebra. □

We need some preparation for the other cases. For effective Cartier divisors D, D' we define $\epsilon(D, D') = \max \{ord_V(D) \cdot ord_V(D') | \text{codim(V,X)} = 1\}$. Since they are effective, it is clear that they meet properly iff $\epsilon(D, D') = 0$. Then let $D \cap D'$ be the intersection scheme of D and D'. This is locally defined by the ideal (a, a') with notation as above. Let $\pi : \tilde{X} \to X$ be the blow up of X along $D \cap D'$ with exceptional divisor $E = \pi^{-1}(D \cap D')$. Since the local equations of π^*D and π^*D' are divisible by the local equation of E, we get that $pi^*D = E + C, \pi^*D' = E + C'$ for effective Cartier divisors C, C' on $\tilde{X}$. We have the following useful lemma:

Lemma 1.31. *(a) C and C' are disjoint; and (b) If $\epsilon(D, D') > 0$ then $\epsilon(C, E)$ and $\epsilon(C', E)$ are strictly smaller than $\epsilon(D, D')$.*

Proof. Since the assertions are local (disjointness is clearly and the other involves calculations in local rings so is local), we may assume $X = \text{Spec } A$ and $D \cap D'$ is given by the ideal $I = (a, a')$. Then in this patch we can take $\tilde{X} = \text{Proj } \oplus I^n$. From the graded algebra surjection $A[S, T] \to \oplus I^n$ with $S \mapsto a, T \mapsto a'$, we get a closed immersion $\tilde{X} \to X \times \mathbb{P}^1$ which commutes with the projection down to X. Moreover, $\tilde{X}$ is, by definition of this homomorphism, contained in the subscheme defined by the vanishing of $a'S - aT$. Let $\mathcal{O}(1)$ be the line bundle on $\tilde{X}$ associated with this embedding (i.e. the pull back of the tautological bundle on $\mathbb{P}^1$ with sections s, t corresponding to S, T, respectively. We show that C (resp. C') is the zero-scheme of the section s (resp. t). Since we have $a's = at$, on the patch $s \neq 0$, we get $a' = (t/s)a$ and thus $\pi^* D = E$ on this patch since $(a, a') = (a, (t/s)a) = (a)$ as ideals in this affine patch of $\tilde{X}$. But in the affine patch given by $t \neq 0$, $a = (s/t)a'$ so $(a) = ((s/t)a') = (s/t) \cap (a, a')$ and thus $\pi^* = E + Z(s)$. Thus overall $\pi^* D = E + Z(s)$ and similarly $\pi^* D' = E + Z(t)$, which determines explicitly what C and C' are. Since $Z(s) \cap Z(t) = \varnothing$, this proves (a)

From the description above it is clear that $C \subset X \times \{0\}, C' \subset X \times \{\infty\}$ map isomorphically onto D, D' , respectively. Thus if $\tilde{V}$ is a codimension one subvariety of $\tilde{X}$ contained in $C \cap E$, then it is a codimension one subvariety in $X \times \{0\}$ which is thus mapped isomorphically to a codimension one subvariety V of X by the first projection (which coincides with the blow-up morphism), and moreover this image is obviously contained in $D \cap D'$ since $V \subset E$. By the projection formula, $[D] = D \cdot [X] = D \cdot \pi_*[\tilde{X}] = \pi_*(\pi^* D \cdot [\tilde{X}]) = \pi_*([E + C])$. Thus $ord_V[D] = ord_V \pi_*[E] + ord_V \pi_*[C]$. But since there could be other codimension 1 subvarieties mapping onto V from inside C and E which would increase the coefficient of $[V]$ inside the pushforward, we must have $ord_V[D] \geq ord_{\tilde{V}}[E] + ord_{\tilde{V}}[C]$, and similarly if $\tilde{V} \subset C' \cap E$. Now suppose the result were false; then $\epsilon(C, E) \geq \epsilon(D, D') > 0$, and we can choose $\tilde{V} \subset C \cap E$ such that $ord_{\tilde{V}} C \cdot ord_{\tilde{V}} E = \epsilon(C, E)$. Then $\epsilon(D, D') \geq ord_V D \cdot ord_V D' \geq (ord_{\tilde{V}} E + ord_{\tilde{V}} C)(ord_{\tilde{V}} E')$, where we note that even if $\tilde{V}$ is not contained in C' the same argument above gives the required inequality. But then we get that $\epsilon(D, D') \geq (ord_{\tilde{V}} E)^2 + \epsilon(C, E)$, a contradiction. ☐

We also need the following lemma which follows from the projection formula and the validity of the theorem on $\tilde{X}$:

Lemma 1.32. *If D, D' are Cartier divisors on X and $\pi : \tilde{X} \to X$ is a proper birational morphism of varieties with $\pi * D = B \pm C, \pi^* D' = B' \pm C'$ for Cartier divisors s.t. $|B| \cap |C| \subset \pi^{-1}(|D|), |B'| \cap |C'| \subset \pi^{-1}(|D'|)$, and the theorem holds for each of the individual pairs on $\tilde{X}$, then the theorem holds for (D, D') on X.*

We are now ready to return to the proof of the theorem:

Proof. Case 2: Suppose D, D' are effective. If $\epsilon(D, D') = 0$, then we're done by Case 1. If not then by Lemma 1.31 can blow-up and use induction upstairs to get the validity of the theorem there. By Lemma 1.32 it is true downstairs as well.

Case 3: Suppose D' is effective. Let $\mathcal{J}$ be the following ideal: locally on $U = \text{Spec } A$ where D has local equation d, $\mathcal{J}(U) = \{a \in A | ad \in A\}$, i.e. the denominator ideal of d. A local calculation shows that $\pi^* D = C - E$ for some effective Cartier divisor C, where $\pi : \tilde{X} \to X$ is the blow-up along $\mathcal{J}$ with exceptional divisor E. Then we can apply Case 2 to the pairs $(C, \pi^* D')$ and $(E, \pi^* D')$ and then apply Lemma 1.32 to complete the proof.

Case 4: Now let D, D' be arbitrary. We blow up by the ideal of denominators of D as above. Then we apply Case 3 to $(C, \pi * D')$ and (E, π^*) and use Lemma 1.32 to finish. $\square$

As an easy consequence of this theorem we get the somewhat obvious corollary:

Corollary 1.33. *For $\alpha \sim 0$ we have $D \cdot \alpha = 0$*

Proof. Clearly we may suppose that $\alpha = [div(r)]$ for $r \in K(V)^*$. Moreover, by the projection formula we may suppose $V = X$ and D is Cartier. Then $D \cdot [div(r)] = div(r) \cdot [D] = 0$ from the commutativity theorem and (e) of Proposition 1.28. $\square$

By the projection formula and the commutativity theorem, we get another obvious corollary:

Corollary 1.34. $D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha)$.

Definition 1.35. If we intersect n divisors $D_1, ..., D_n$ with an n-cycle α and $|D_1| \cap ... \cap |D_n| \cap |\alpha|$ is complete (for example if X is proper), we define the **intersection number** $(D_1 \cdot ... \cdot D_n \cdot \alpha)_X = \int_Y D_1 \cdot ... \cdot D_n \cdot \alpha$. Similarly we can define the same thing for a homogeneous polynomial.

Example 1.36. *We explore now the remark Fulton makes about how if a pseudo-divisor has trivial line bundle, then intersecting with it defines a genuine cycle and how this relates to commutativity. Let $\pi : X \to \mathbb{A}^2$ be the blow-up at the origin and D, D' the Cartier divisors on the blow-up defined by inverse images of the x-axis and y-axis,respectively. That is $D = E + C, D' = E + C'$, where C, C' are the proper transforms of the x-axis and y-axis respectively. Then according to the definitions $D \cdot [D'] = D \cdot ([E] + [C']) = 0 + D \cdot [C']$ since $E \subset |D|$ so is set to zero. $D \cdot [C']$ gives the point of intersection between the proper transform of the y-axis and E since it has no intersectoin with C as follows from Lemma 1.31. Similarly, $D' \cdot [D]$ gives the point of intersection of C and E. Notice that these are two different cycles on $D \cap D' = E$ but which are equal classes in the Chow group of E.*

Example 1.37. *Let $X \subset \mathbb{P}^3$ be the singular cone defined by the equation $z^2 = xy$. Let D be the Cartier divisor on X defined by $x = 0$. Let $[L]$ be the line $x = z = 0$ and $[L']$ be the line $y = z = 0$, and $P = (0 : 0 : 0 : 1)$. Then clearly $[D] = 2[L]$ and $D \cdot [L'] = [P]$ since the way we defined the intersection product we forget about the embedding of L' in X. By commutativity, if there were a Cartier divisor D' with Weil divisor $[L']$, then $[P] = D \cdot [L'] = D' \cdot [D] = 2D' \cdot [L] = 2[P]$, a contradiction both on the level of cycles and on classes of cycles since X is proper and thus points cannot be rationally equivalent to zero.*

Example 1.38. *Suppose we intersect n Cartier divisors $D_1, ..., D_n$ on an n-dimensional variety X which intersect in a finite set and whose local equations form a regular sequence in each $\mathcal{O}_{P,X}$ for P in this finite set, then from properties of Hilbert-Samuel polynomials and the definition of the intersection number we get $(D_1 \cdot ... \cdot D_n)_X = \deg[D_1 \cap ... \cap D_n]$.*

1.2.4 Chern classes of a Line Bundle

Definition 1.39. We define the action of the (first) Chern class of a line bundle L on a subvariety V of a scheme X as follows: restrict L to V and consider a Cartier divisor C representing the pseudo-divisor $(L|_V, V, s)$, which is unique up to linear equivalence. Then define $c_1(L) \cap [V] = [C]$, the Weil divisor associated to C. We extend this by linearity to a homomorphism of $c_1(L)\cap : Z_k(X) \to A_{k-1}(X)$. Moreover if $L = \mathcal{O}_X(D)$ for some divisor then clearly $c_1(L)\cap\alpha = D\cdot\alpha$.

We have the following proposition which follows directly from Proposition 1.28 and the commutativity theorem.

Proposition 1.40. *(a) If $\alpha \sim 0$, then $c_1(L) \cap \alpha = 0$.*

(b) For two line bundles L, L', we have $c_1(L)\cap(c_1(L')\cap\alpha) = c_1(L')\cap(c_1(L)\cap \alpha)$.

(c) For f proper, $f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*(\alpha)$.

(d) For f flat, $c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha)$.

(e) $c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$ and $c_1(L^{-1}) = -c_1(L) \cap \alpha$.

Example 1.41. *Clearly $c_1(\mathcal{O}(1)) \cap [L^k] = [L^{k-1}]$ in for $\mathbb{P}^n$. Because of this we can define the degree of a k-cycle in $\mathbb{P}^n$ as the unique d such that $\alpha \sim d[L^k]$, or equivalently* $\deg(\alpha) = \int_{\mathbb{P}^n} c_1(\mathcal{O}(1))^k \cap \alpha$.

1.2.5 Gysin Map for Divisors

Definition 1.42. We can define the Gysin map $i^* : Z - k(X) \to Z_{k-1}(D)$ for effective Cartier divisor D with inclusion $i : D \to X$ as $i^*(\alpha)) = D \cdot \alpha$.

We have the following analogous properties for the Gysin map whose proofs are similar to all of the above propositions:

Proposition 1.43. *(a) If $\alpha \sim 0$ then $i^*\alpha = 0$.*

(b) For $\alpha \in Z_k(X)$, $i_*i^*(\alpha) = c_1(\mathcal{O}_X(D)) \cap \alpha$.

(c) If $\alpha \in Z_k(D)$, $i^*i_*(\alpha) = c_1(N) \cap \alpha$.

(d) If X is purely n-dimensional then $i^*[X] = [D]$ (this one isn't actually like the above but it follows from the algebraic lemma of section 1)

(e) $i^*(c_1(L) \cap \alpha) = c_1(L) \cap i^*(\alpha)$.

We can now discuss some interesting geometric corollaries of this result.

Corollary 1.44. *For a line bundle L, $p^* : A_k(X) \to A_{k+1}(X)$ is an isomorphism.*

Proof. Let $i : X \to L$ be the embedding induced by the zero-section. Then it's clear that $i^*(p^*\alpha) = \alpha$. So p^* is injective. Since it was already surjective we're done. □

Corollary 1.45. *If X is a closed subscheme of $\mathbb{P}^n$ and X' is its projective cone in $\mathbb{P}^{n+1}$, then $A_0(X') \cong \mathbb{Z}$ and $A_k(X') \cong A_{k-1}(X)$ for $k > 0$.*

Proof. Let P be the vertex of the cone. Then $X' - P$ is a line bundle over X. Thus from the exact sequence $A_k(P) \to A_k(X') \to A_k(X' - P) \to 0$, we see that for $k = 0$ $A_0(X' - P) = 0$ since $X' - P$ is a line bundle and along each $\mathbb{A}^1$ we can find a rational function vanishing precisely at a single point, so $A - K(P) \to A_k(X')$ is surjective. It is also injective because X' is complete. For $k > 0$ we get that $A_k(P) = 0$, so $A_k(X') \cong A_k(X' - P) \cong A_{k-1}(X)$, this latter isomorphism from Corollary 1.44. □

Corollary 1.46. *(a) Let L be a line bundle on X and $L - \{0\}$ the complement of the zero section with projection $\eta : L - \{0\} \to X$. Then*

$$A_{k+1}(X) \to A_k(X) \to A_{k+1}(L - \{0\}) \to 0,$$

where the first map is the chern class of L and the second is flat pull-back.

(b) Let X be a closed subscheme of $\mathbb{P}^n$ and $V \subset \mathbb{A}^{n+1}$ be the affine cone of X. Then $A_0(V) = 0$ and for $k > 0$ there is an exact sequence

$$A_k(X) \to A_{k-1}(X) \to A_k(V) \to 0,$$

where the first map is the chern class of $\mathcal{O}_X(1)$.

Proof. For part (a), let $i : X \to L$ be the inclusion induced by the zero-section and $j : L - \{0\} \to L$ be the open immersion. Then we have the excision exact sequence

$$A_{k+1}(X) \to A_{k+1}(L) \to A_{k+1}(L - \{0\}) \to 0,$$

where the first map is i_* and the second is j^*. Now notice that η factors as $p \circ j$, so we may factor $\eta^* = j^* \circ p^*$, where from (1.45) we know that p^* is an isomorphism with inverse i^*. Moreover $i^* \circ i_* = c_1(N)\cap$, but the normal bundle of the zero-section embedding is just L. This allows us to insert $A_k(X)$ into $A_{k+1}(L)$'s spot without effecting exactness. This above argument also shows why the maps are as stated.

For (b), notice that from a local calculation, if we blow-up $\mathbb{A}^{n+1}$ at the origin, then the blow-up becomes the total space of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^n$ and thus on X. Since away from the origin this is an isomorphism we see that $V - P$ is isomorphic to the total space of $\mathcal{O}_X(-1) - \{0\}$, where P is the origin. We use this to construct the exact sequence. First for $k = 0$ we notice that any point of V lies on some line. This line allows us to construct a rational function vanishing only at that point. Thus $A_0(V) = 0$. From the excision exact sequence, we have $A_k(P) \to A_k(V) \to A_k(V - P) \to 0$, which gives $A_k(V) \cong A_k(V - P)$ for $k > 0$. So we in fact can restrict ourselves to $V - P$. Then from the above comments and part (a) we get the desired exact sequence after we multiply everything by -1 to get $c_1(\mathcal{O}(1))$ instead of $\mathcal{O}(-1)$. □

Example 1.47. *We present here an example that is important later on with deformation to the normal cone. Let α be a k-cycle on $X \times \mathbb{P}^1$, and let i_0, i_∞ be the embeddings of X at 0 and ∞, respectively. Then we get that $i_0^*\alpha = i_\infty^*\alpha$. Clearly it suffices to check this on subvarieties. If a subvariety V does not* *project dominantly to $\mathbb{P}^1$ then the closure of its image is an irreducible closed subset of $\mathbb{P}^1$ which must be a point. Thus V is contained in a fibre. If this fibre happens to be 0 or ∞ then I can move it to a different one since points are rationally equivalent on $\mathbb{P}^1$ and rational equivalence is preserved under flat pull back. Then it becomes clear that both sides of the equation are zero since the schemes have no intersection. If V is projected dominantly, then the statement is that $[V(0)] = [V(\infty)]$ which is true in the Chow group by the alternative definition of rational equivalence.*

1.3 Vector bundles and Chern classes

1.3.1 Segre classes of Vector Bundles

Fulton defines Chern classes via some more geometric "cohomology" classes called Segre classes, so we define these first.

Definition 1.48. Given a rank $e+1$ vector bundle E, let $\mathcal{E} = \mathcal{O}(E)$, and $P = P(E) = \mathbb{P}(\mathcal{E}^\vee)$, the projective bundle of lines in E. Let p be the natural projection from P to X, $\mathcal{O}(1) = \mathcal{O}_E(1)$ the standard line bundle on P and $\mathcal{O}(-1)$ the tautological subbundle of p^*E, its dual. We define the i^{th} Segre homomorphism $s_i(E) \cap \alpha = p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^*\alpha)$. Since p is both proper and flat, both the push-forward and pull-back are defined.

We have the following proposition which allows us to effectively deal with Segre classes.

Proposition 1.49. *(a) $s_i(E) \cap \alpha = 0$ for $i < 0$ and $s_0(E) \cap \alpha = \alpha$.*
(b) $s_i(E) \cap (s_j(F) \cap \alpha) = s_j((F) \cap (s_i(E) \cap \alpha)$.
(c) For $f : X' \to X$ proper, $f_*(s_i(f^*E) \cap \alpha)) = s_i(E) \cap f_*(\alpha)$.
(d) For $f : X' \to X$ flat, $s_i(f^*E) \cap f^*\alpha = f^*(s_i(E) \cap \alpha)$.
(e) If E is line bundle, $s_1(E) \cap \alpha = -c_1(E) \cap \alpha$.

Proof. (c) and (d) will follow from the projection formula and commutativity of pull-back and push-forward in a fibre square. We form the fibre square,

$$\begin{array}{ccc} P(f^*E) & \xrightarrow{f'} & P(E) \\ {\scriptstyle p'}\downarrow & & {\scriptstyle p}\downarrow \\ X' & \xrightarrow{f} & X \end{array},$$

where $\mathcal{O}_{f^*E}(1) = f'^*\mathcal{O}_E(1)$ by construction. Then

$$f_*(s_i(f^*E) \cap \alpha) = f_*p'_*(c_1(\mathcal{O}_{f^*E}(1))^{e+i} \cap p'^*\alpha) = p_*f'_*(c_1(f'^*\mathcal{O}_E(1))^{e+i} \cap p'^*\alpha),$$

by functoriality. By the projection formula, this is $p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap f'_*p'^*\alpha)$. Using the commutativity of pull-back and push-forward around a fibre square we get this is just

$$p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap p^*f_*\alpha) = s_i(E) \cap f_*\alpha,$$

which is what we want for (c). The proof of (d) is similar. For (a), it clearly suffices to prove this for $\alpha = [V]$. By the projection formula, we may suppose $V = X$. Then $A_{k-i}(X) = 0$ for $i < 0$ so this proves the first part. For the second part of (a), we have $s_0(E) \cap [X] = p_*(c_1(\mathcal{O}_E(1))^e \cap [P])$. We now show that this is $[X]$. By part (d), we may restrict to an open set of X upon which E is trivial, so we may assume it's trivial from the outset. Then $P = X \times \mathbb{P}^e$, so from a previous example we have $c_1(\mathcal{O}_E(1))^e \cap [X \times \mathbb{P}^e]) = [X \times pt]$. Projecting down to X gives $[X]$ as claimed.

For (b) we form the fibre square,

$$\begin{array}{ccc} Q & \xrightarrow{p'} & P(F) \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ P(E) & \xrightarrow{p} & X \end{array}$$

. Let $f+1$ be the rank of F. Then

$$s_i(E) \cap (s_j(F) \cap \alpha) = p_*(c_1(\mathcal{O}_E(1))^{e+i}) \cap p^*(q_*(c_1(\mathcal{O}_F(1))^{f+j} \cap q^*\alpha))).$$

Using the commutativity of push-forward and pull-back around the fibre square, the projection formula and flat pull-back formula, and the corresponding commutativity for first chern classes gives the result.

For (e), we note that $P = X$ and $\mathcal{O}_E(-1) = E$, in which case $\mathcal{O}_E(1) = E^{-1}$, so the result follows from that fact that $c_1(L^{-1}) \cap \alpha = -c_1(L) \cap \alpha$. □

Corollary 1.50. *$p^* : A_k(X) \to A_{k+e}(P(E))$ is a split injection.*

Proof. From part (a) of the above proposition we have that $\beta \mapsto p_*(c_1(\mathcal{O}_E(1))^e \cap \beta)$ is an inverse. □

1.3.2 Chern Classes

Definition 1.51. Define a formal power series $s_t(E) = \sum_{i=0}^{\infty} s_i(E)t^i$. Define $c_t(E) = s_t(E)^{-1}$, as a formal power series. The i^th chern class, $c_i(E)$ is defined to be the coefficient of t^i in $c_t(E)$. Thus $c_n(E) = -s_1(E)c_{n-1}(E) - s_2(E)c_{n-2}(E) - ... - s_n(E)$, where the first two terms are $c_0(E) = 1, c_1(E) = -s_1(E)$. We define the total Segre class and total Chern class by setting $t = 1$. These sums are finite since the take a k-cycle to a class in $A_{k-i}(X)$ and thus vanish for $i > \dim(X)$.

The big theorem about Chern classes is:

Theorem 1.52. *(a) $c_i(E) = 0$ for $i >$ rank (E).*

(b) $c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha)$.

(c) For $f : X' \to X$ proper, $f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*\alpha$.

(d) For $f : X' \to X$ flat, $c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha)$.

(e) For an exact sequence $0 \to E' \to E \to E'' \to 0$, $c_t(E) = c_t(E')c_t(E'')$, that is $c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'')$. This is known as the Whitney sum formula.

(f) If E is line bundle, X is a variety, and D is Cartier divisor with $\mathcal{O}_X(D) = E$, then $c_1(E) \cap [X] = [D]$.

Proof. Properties (b),(c),(d),(f) all follow from previous results about Segre classes. The proof of the others require the splitting construction. □

Splitting construction We show that for a finite collection of vector bundles S on a scheme X, there is a flat morphism $f : X' \to X$ such that $f^* : A_*(X) \to A_*X'$ is injective, and for each vector bundle $E \in S$, $f^*E = E_r \supset E_{r-1} \supset ... \supset E_1 \supset E_0 = 0$, with line bundle quotients $L_i = E_i/E_{i-1}$. The construction goes as follows: We take one bundle E and construct X' by induction on the rank of E. Let $p : P(E) \to X$ be the projection. Then p^* is injective by (1.50), and p^*E has a subbundle $\mathcal{O}_E(-1)$ of rank one. Take the quotient bundle E'. By induction we can construct $q : X' \to P$ with q^* injective and q^*E' filtered. Then $f = pq$ still has f^* injective and we get a filtration of f^*E. We can repeat this for a finite number of bundles.

Lemma 1.53. *Assume that E is filtered as in the splitting construction with line bundle constructions $L_1, ..., L_r$. Let s be a section of E and $Z = Z(s)$. Then for any k-cycle α on X, there is a $(k-r)$-cycle β on Z with $\prod_{i=1}^r c_1(L_i) \cap \alpha = \beta \in A_{k-r}(X)$. Thus if s is nowhere vanishing then $\prod_{i=1}^r c_1(L_i) = 0$.*

Proof. The section s determines a section $\overline{s}$ in the quotient line bundle $L_r = E/E_{r-1}$. If $Y = Z(\overline{s})$, then $(L_r, Y, \overline{s})$ determines a pseudo-divisor D_r on X. So taking $D_r \cdot \alpha \in A_{k-1}(Y)$, we see $c_1(L_r) \cap \alpha = j_*(D_r \cdot \alpha)$, where j is the inclusion of Y in X. By the projection formula

$$\prod_{i=1}^{r} c_1(L_i) \cap \alpha = \prod_{i=1}^{r-1} c_1(L_i) \cap j_*(D_r \cdot \alpha) = j_*(\prod_{i=1}^{r-1} c_1(L_i) \cap (D_r \cdot \alpha)).$$

By induction on r, we get β as required. □

Lemma 1.54. *If a vector bundle E on a scheme has a filtration $E = E_r \supset E_{r-1} \supset ... \supset E_0 = 0$ by subbundles with quotient line bundles $L_i = E_i/E_{i-1}$, then $c_t(E) = \prod_{i=1}^r (1 + c_1(L_i)t)$.*

Proof. Let $p : P(E) \to X$ be the projective bundle. Then $\mathcal{O}(-1)$ is the universal subbundle of p^*E which gives a nowhere vanishing section of $p^*E \otimes \mathcal{O}_E(1)$ by tensoring the inclusion by $\mathcal{O}_E(1)$. Now the filtration on E pulls back to a filtration of p^*E which we tensor with $\mathcal{O}_E(1)$ to give quotient line bundles $p^*L_i \otimes \mathcal{O}(1)$. By the previous lemma $\prod_{i=1}^r c_1(p^*L_i \otimes \mathcal{O}_E(1)) = 0$ since the section is nowhere vanishing. Now for notational convenience let $\zeta = c_1(O(1))$, σ_i (resp. $\tilde{\sigma_i}$) the ith symmetric polynomial in $c_1(L_1), ..., c_1(L_r)$ (resp. their pull backs to P). Then we have $c_1(p^*L_i \otimes \mathcal{O}(1)) = \zeta + c_1(p^*L_i)$. By the above equation on these products we get that $\zeta^r + \tilde{\sigma_1}\zeta^{r-1} + ... + \tilde{\sigma_r} = 0$. Letting $e = r-1$ and multiplying by ζ^{i-1} we get that for all $\alpha \in A_*(X)$, $p_*(\zeta^{e+i} \cap$

$p^*\alpha) + p_*(\tilde{\sigma_1}\zeta^{e+i-1} \cap p^*\alpha) + ... + p_*(\tilde{\sigma_r}\zeta^{i-1} \cap p^*\alpha) = 0$. But this just means, from the definition of the Segre class and the projection formula, that

$$s_i(E) \cap \alpha + \sigma_1 s_{i-1}(E) \cap \alpha + ... + \sigma_r s_{i-r}(E) \cap \alpha = 0.$$

Using this vanishing for all i in writing out the expression $(1 + \sigma_1 t + ... + \sigma_r t^r)s_t(E)$, we find that this expression is just 1. But this implies that indeed $c_t(E) = \prod_{i=1}^r (1 + c_1(L_i)t)$. □

Proof. Now we can prove part (a) and (e) of the theorem. For (a), If $f : X' \to X$ is as in the splitting construction, then $f^*(c_i(E) \cap \alpha) = c_i(f^*E) \cap f^*\alpha$. But then f^*E has a filtration as in the lemma so $c_i(f^*E) = 0$ for $i >$ rank (E) by the lemma. Thus $f^*(c_i(E) \cap \alpha) = 0$, and thus $c_i(E) \cap \alpha = 0$ since f^* is injective.

For (e), given an exact sequence of vector bundles, $0 \to E' \to E \to E'' \to 0$, we can take an $f : X' \to X$ as in the splitting construction that gives a filtration for f^*E', f^*E'' with f flat and f^* injective. Suppose f^*E' has line bundle quotients L'_i and f^*E'' has line bundle quotients L''_j. Then this induces a filtration on f^*E with line bundle quotients L'_i, L''_j. Then by Lemma 1.54, we get that $c_t(f^*E) = c_t(f^*E')c_t(f^*E'')$. By the injectivity of f^* we get the same result on X. □

Remark 1.55. When we write $c_t(E) = \prod_{i=1}^r (1 + \alpha_i t)$, we call the α_i the **Chern roots** of E. We saw from the proofs above that the Chern class $c_i(E)$ is the i-th elementary symmetric polynomial in the Chern roots, and by the fundamental theorem of symmetric polynomials we get that any symmetric polynomial in the Chern roots is a polynomial in the Chern classes. This is an important fact when dealing with chern classes.

We can use the theorem and the splitting construction to calculate how Chern classes change under various operations on vector bundles.

Dual bundles For example, given a filtration $0 \subset E_1 \subset ... \subset E_r = E$ with line bundle quotients $E_i/E_{i-1} = L_i$ and chern roots $\alpha_1, ..., \alpha_r$, then it's not hard to see that $E^\vee$ has a filtration with quotients $L^\vee_{r-i}$. Then by the result for chern classes of line bundles we get that the chern roots of $E^\vee$ are $-\alpha_1, ..., -\alpha_r$ which gives $c_i(E^\vee) = (-1)^i c_i(E)$.

Tensor products Using an induction argument, the splitting principle, the Whitney sum formula, and the result on chern classes of tensors of line bundles, we get that the chern roots of $E \otimes F$ are the sums $\alpha_i + \beta_j$ for all $1 \le i \le r, 1 \le j \le s$, where α_i, β_j are the chern roots of E, F, respectively. For $F = L$ a line bundle, we get that $c_t(E \otimes L) = sum_{i=0}^r t^i c_t(L)^{r-i} c_i(E)$.

Exterior powers Using the splitting principle we may assume E has a filtration by subbundles with line bundle quotients. Let $\alpha_1, ...\alpha_r$ be the Chern roots of E. We show that $c_t(\bigwedge^p E) = \prod_{i_1 < ... < i_p} (1 + (\alpha_{i_1} + ... + \alpha_{i_p})t)$. If we let L be the first subbundle in the filtration then it's a line bundle, so we have an exact sequence $0 \to L \to E \to E' \to 0$. Some basic linear algebra gives $0 \to \bigwedge^{p-1} E' \otimes L \to \bigwedge^p E \to \bigwedge^p E' \to 0$. Using the Whitney sum formula we get $c_t(\bigwedge^p E) = c_t(\bigwedge^{p-1} E' \otimes L)c_t(\bigwedge^p E')$, and using induction on both p and the rank of the bundle we get $c_t(\bigwedge^p E) = \prod_{1<i_1<...<i_{p-1}}(1 + (\alpha_{i_1} + ... + \alpha_{i_{p-1}} + \alpha_1)t) \prod_{1<i_1<...<i_p}(1 + (\alpha_{i_1} + ... + \alpha_{i_p})t)$, which gives the desired result when we note that L has chern root α_1 and E' has the other chern roots.

Symmetric powers A similar calculation as above involving an analogous exact sequence shows that the Chern roots of $Sym^m(E)$ are exactly $\sum m_1\alpha_1 + ... + m_r\alpha_r$ over r-tuples $(m_1, ..., m_r)$ with $\sum m_i = m$.

Chern character This is an important invariant of a vector bundle E and becomes very important in the statement and proof of GRothendieck-Riemann-Roch. We define

$$ch(E) = \sum_{i=1}^{r} \exp(\alpha_i)$$

,where α_i are the Chern roots of E. Writing out the first few terms of the chern character, we get

$$ch(E) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + ...,$$

where $c_i = c_i(E)$. The nth term is $p_n/n!$, where p_n is determined inductively by

$$p_n - c_1p_{n-1} + c_2p_{n-2} - ... + (-1)^{n-1}c_{n-1}p_1 + (-1)^n nc_n = 0.$$

From the above calculations with chern roots we see that $ch(E) = ch(E') + ch(E'')$ for any exact sequence $0 \to E' \to E \to E'' \to 0$, and $ch(E \otimes E') = ch(E) \cdot ch(E')$. The Chern class of an n-dimensional variety X is defined by $c(T_X) \cap [X]$. The **Euler characteristic** is defined to be $\int_X c_n(T_X)$.

Todd Class This is another invariant that appears in the GRR theorem. We define

$$td(E) = \prod_{i=1}^{r} Q(\alpha_i),$$

where $Q(x) = \frac{x}{1-e^{-x}}$ and again the α_i are the chern roots. The first few terms are

$$td(E) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}(c_1c_2) + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + ...$$

For an exact sequence again we get $td(E) = td(E') \cdot td(E'')$. There is a beautiful relation between the Chern character and Todd class of a vector bundle:

$$\sum_{p=0}^{r} (-1)^p ch(\bigwedge^p E^\vee) = c_r(E) td(E)^{-1},$$

whose proof is just a simple calculation maniulating both sides. When talking about a non-singular variety we define the **Todd class** of X to be $td(T_X)$.

Example 1.56. *In analogy with the corresponding result for chern classes of line bundles we have* $c(E) \cap \alpha) \times (c(F) \cap \beta) = c(p^*E \oplus q^*F) \cap (\alpha \times \beta)$

Now that we have seen how to manipulate chern classes, we can apply them in actual geometric situations. The base calculation for everything is the following example:

Example 1.57. *We have the famous exact sequence on $\mathbb{P}^n$,*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0.$$

Using the Whitney sum formula we get that $c(T_{\mathbb{P}^n}) = (1+H)^{n+1}$, where $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. In line with Grothendieck's philosophy of treating things relatively, we get the following slightly longer exact sequence for X a nonsingular variety, E a vector bundle, and $p : P(E) \to X$ the projection:

$$0 \to \mathcal{O}_{P(E)} \to p^*E \otimes O_E(1) \to T_{P(E)} \to p^*T_X \to 0.$$

This exact sequence comes partially from the relative version of the short exact sequence above

$$0 \to \mathcal{O}_{P(E)} \to p^*E \otimes \mathcal{O}_E(1) \to T_{P(E)/X} \to 0,$$

whose proof just comes from patching together the same proof over a random affine scheme from Hartshorne. We also have the stanard cotagent right exact sequence

$$p^*\Omega_X \to \Omega_{P(E)} \to \Omega_{P(E)/X} \to 0,$$

in which every term is actually locally free, and moreover in this specific case the map on the left is injective as one can check locally on $P(E)$ where we can treat X is a regular affine ring and $P(E)$ is just $X \times \mathbb{A}^n$. So dualizing (which is still right exact since everything is locally free). we get an exact sequence

$$0 \to T_{P(E)/X} \to T_{P(E)} \to p^*T_X \to 0,$$

which we can combine with the above to get the four-term exact sequence. Taking Chern classes we get the formula,

$$c_t(T_{P(E)}) = c_t(p^*T_X) \cdot c_t(p^*E \otimes \mathcal{O}_E(1)).$$

Moreover, we can use the adjunction exact sequence to relate the Chern classes of subvarieties to that of the ambient variety:

Example 1.58. *Given a smooth subvariety $X \subset Y$ of a smooth variety Y of codimension d with normal bundle N, we get the adjunction short exact sequence,*

$$0 \to T_X \to T_Y|_X \to N \to 0,$$

Taking Chern classes gives $c(T_X) = c(T_Y|_X)/c(N)$, so its chern class is totally dependent on the embedding. For example, if X is the complete intersection of divisors D_i in $Y = \mathbb{P}^n$ with $\deg D_i = n_i$, then $c(T_X) = (1+h)^{n+1}/\prod_{i=1}^d (1+n_i h)$.

We have the following useful proposition which is important for Degeneracy loci.

Proposition 1.59. *Given a vector bundle E of rank r and s a section of E, then for any $\alpha \in A_k(X)$ there is a $\beta \in A_{k-r}(Z(s))$ such that $c_r(E) \cap \alpha = \beta$. Moreover, if X is pure dimensional and s is a regular section, then $Z(s)$ is purely $(n-r)$-dimensional and $c_r(E) \cap [X] = [Z]$*

Proof. If we use the splitting principle, then we may assume E is filtered. Then $c_r(E)$ is the product mentioned in the lemma so the existence of β just follows from there. If X is pure dimensional and s is regular, then $Z(s)$ is purely $(n-r)$-dimensional from standard commutative algebra about regular sequences. The equality of cycles follows from the splitting principle and Lemma 1.14. □

We have the following important use of Chern classes which generalizes the classical Riemann-Hurwitz formula:

Example 1.60. *Let $f : X \to Y$ be a morphism of smooth n-dimensional varieties. Let $R(f)$ be the set where the induced map of tangent spaces is not an isomorphism. This is given a scheme structure as the vanishing of the Jacobian of the map f, i.e. the zero-scheme of the map $\bigwedge^n df : \bigwedge^n T_X \to \bigwedge^n f^*T_Y$, or equivalently the zero-scheme of the induced section of $\bigwedge^n f^*T_Y \otimes \bigwedge^n T_X^\vee$. We see that $R(f)$ has codimension at most 1. If $R(f) \neq X$ then it has codimension 1, and since X is smooth and thus CM, we get that the induced section must be regular from a commutative algebra result (Fulton's A.7.1). Thus by the previous proposition, $[R(f)] = (c_1(f^*T_Y) - c_1(T_X)cap[X]$. Now if $n = 1$, we get upon taking degrees that* $\deg R(f) = \int_X c_1(f^*T_Y) \cap [X] - \int_X c_1(T_X) \cap [X] = \int_Y f_*(c_1(f^*T_Y) \cap [X]) - \int_X c_1(T_X) \cap [X] = \int_Y c_1(T_Y) \cap f_*([X]) - (2 - 2g_X)$ *by functoriality of push-forwards and the projection formula for Chern classes. But* $f_*([X]) = \deg(f)[Y]$, *so in fact we see the first term is* $\deg(f)(2 - 2g_Y)$. *Rearranging, we get the classical Riemann-Hurwitz formula*

$$2g_X - 2 = \deg(f)(2g_Y - 2) + \deg(R(f)).$$

Finally, we present an example of how Chern classes

1.3.3 Rational Equivalence on Bundles

Theorem 1.61. *(a) The flat pull-back $\pi^* : A_{k-r}(X) \to A_k(E)$ is an isomorphism for a vector bundle E of rank $r = e + 1$, π the projection.*

(b) Each element $\beta in A_k P(E)$ has a unique expression $\beta = \sum_{i=0}^e c_1(\mathcal{O}_E(1))^i \cap p^*\alpha_i$ for some $\alpha_i \in A_{k-e+i}(X)$. Thus there are canonical isomorphisms

$$\oplus_{i=0}^e A_{k-e+i}(X) \cong A_k(P(E))$$

.

Proof. We first reduce to the case where E is a trivial vector bundle. Define $\theta_E : \oplus_{i=0}^e A_*(X) \to A_*(P(E))$ by $\theta(\oplus \alpha_i) = \sum_{i=0}^e c_1(\mathcal{O}(1))^i \cap p^*\alpha_i$. To show that this map is surjective and to show part (a) (note that we have already shown surjectivity of that map in the chapter on rational equivalence), we may use the excision exact sequence and Noetherian induction to reduce to the case that E is trivial. To prove part (b), by induction we just need to show that θ_F is surjective if θ_E is where $F = E \oplus 1$.

To do this we let $P = P(E)$, $Q = P(F) = P(E \oplus 1)$, $q : Q \to X$ the projection. We have the following commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{i} & Q & \xleftarrow{j} & E \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle \pi}\downarrow V \\ X & \xrightarrow{id} & X & \xleftarrow{id} & X \end{array}$$

, where Q is the projective competion of E and P is the hyperplane at infinity, its complement. By the excision exact sequence applied again, we get the following commutative diagram with exact top row:

$$\begin{array}{ccccccc} A_k(P) & \xrightarrow{i_*} & A_k(Q) & \xrightarrow{j^*} & A_k(E) & \longrightarrow & 0 \\ {\scriptstyle p^*}\uparrow & & {\scriptstyle q^*}\uparrow & & {\scriptstyle \pi^*}\uparrow & & . \\ A_{k-r}(X) & \xrightarrow{id} & A_{k-r}(X) & \xrightarrow{id} & A_{k-r}(X) & & \end{array}$$

We have the following claim: *Claim* $c_1(\mathcal{O}_F(1)) \cap q^*\alpha = i_*p^*\alpha$. Clearly, if suffices to prove this for $\alpha = [V]$, where V is a subvariety. But then since $\mathcal{O}_F(1)$ has a section vanishing precisely on P, we get from the definition of Chern classes that capping $c_1(\mathcal{O}_F(1))$ just intersects with this divisor. Thus we indeed get the equality $c_1(\mathcal{O}_F(1)) \cap [q^{-1}(V)] = [p^{-1}(V)]$.

This allows us to make the calculations we need for the theorem. Indeed, let $\beta \in A_*(Q)$ and by the surjectivity of π^*, we can find $\alpha \in A_*(X)$ such that $j^*\beta = \pi^*\alpha$. Then by commutativity of the above diagrams we get $\beta - q^*\alpha$ is in the kernel of j^* and by exactness of the first row we have $\beta - q^*\alpha \in \mathrm{Im}(i_*)$. Moreover, since we're assuming that θ_E is surjective we may find α_i, s.t.

$$\beta - q^*\alpha = i_*(\sum_{i=0}^{e} c_1(\mathcal{O}_E(1))^i \cap p^*\alpha_i).$$

Now since $i^*\mathcal{O}_F(1) = \mathcal{O}_E(1)$, by the projection formula we get that the right hand side is $\sum_{i=0}^{e} c_1(\mathcal{O}_F(1))^i \cap i_*p^*\alpha_i$. Then using the claim we see that in fact,

$$\beta = q^*\alpha + \sum_{i=0}^{e} c_1(\mathcal{O}_F(1))^{i+1} \cap q^*\alpha_i,$$

i.e. θ_F is surjective. To show that it's injective we just need to show that the expression for β is unique, or equivalently suppose we had a relation $\sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^*\alpha_i = 0$. Let t be the largest index for which $\alpha_t \neq 0$. Then by (1.49a) $0 = p_*(c_1(\mathcal{O}(1))^{e-t} \cap \beta) = \alpha_t$, a contradiction. Thus we have established (b).

For (a), suppose $\pi^*\alpha = 0$ with $\alpha \neq 0$. Then this is just $j^*q^*\alpha = 0$. Again using exactness of the excision sequence, we get that $q^*\alpha = i_*(\sum_{i=0}^{e} c_1(\mathcal{O}_E(1))^i \cap p^*\alpha_i) = \sum_{i=0}^{e} c_1(\mathcal{O}_F(1))^{i+1} \cap q^*\alpha_i$ using the claim. But this contradicts the uniqueness from above. Thus $\alpha = 0$. □

Definition 1.62. If we defonte by s the zero section of the vector bundle E, then we can define $s^* : A_k(E) \to A_{k-r}(X)$ by $s^* = (\pi^*)^{-1}$, since by the theorem π^* is an isomorphism. This is defined, equivalently, by the fact that π^* is surjective and on subschemes V, $s^*[\pi^{-1}(V)] = [V]$.

We can give a somewhat more constructive formula for s^* in terms of higher Chern classes.

Proposition 1.63. *For a vector bundle E or rank r and any $\beta \in A_k(E)$, $s^*(\beta) = q_*(c_r(\zeta) \cap \overline{\beta})$, wehre $q : P(E \oplus 1) \to X$, ζ is the universal rank r quotient bundle of $q^*(E \oplus 1)$, and $\overline{\beta} \in A_k(P(E \oplus 1))$ restricts to β.*

Proof. To see first that such a $\overline{\beta}$ exists, if $\beta = \sum n_i[V_i]$, we take $\overline{\beta} = \sum n_i[\overline{V_i}]$, where $\overline{V_i}$ is the closure of V_i in $P(E \oplus 1)$. For conciseness, let $F = E \oplus 1$, $Q = P(F)$, i the inclusion of $P = P(E)$ in Q, j the inclusion of E in Q. So in terms of just $\overline{\beta}$ we need to show that $\pi^* q_*(c_r(\zeta) \cap \overline{\beta}) = j^*\overline{\beta}$ from the definition of s^*. From the previous thereom and claim we can write $\overline{\beta} = q^*(\gamma) + i_*(\delta)$ for some $\gamma \in A_*(X), \lambda \in A_*(P)$. Now since $j^*q^* = \pi^*$ and $j^*i_* = 0$, it would suffice to prove that (i) $q_*(c_r(\zeta) \cap q^*\gamma) = \gamma$, and (ii) $c_r(\zeta) \cap i_*\delta = 0$. For (i) we use the Whitney sum formula applied to $0 \to \mathcal{O}_F(-1) \to q^*E \oplus 1 \to \zeta \to 0$, to see that $c_r(\zeta) = \sum_{i=0}^{r} c_1(\mathcal{O}_F(1))^i c_{r-i}(q^*E)$. But then

$$q_*(c_r(\zeta) \cap q^*\gamma) = q_*(\sum_{i=0}^{r} c_1(\mathcal{O}_F(1))^i c_{r-i}(q^*E) \cap q^*\gamma),$$

where we can break this last term up as

$$q_*(c_1(\mathcal{O}_F(1))^r \cap q^*\gamma) + \sum_{i=0}^{r-1} q_*(c_1(\mathcal{O}_F(1))^i c_{r-i}(q^*E) \cap q^*\gamma).$$

But by the formula for flat pull back and chern classes, and upon setting $k = r-i$, each term after the sum sign can be wrriten as

$$q_*(c_1(\mathcal{O}_F(1))^{r+(-k)} \cap q^*(c_k(E) \cap \gamma) = s_{-k}(F) \cap (c_k(E) \cap \gamma) = 0$$

by (1.49a). Moreover the first term can be similarly written as $s_0(F) \cap \gamma = \gamma$, which proves (i).

For (ii), we notice that $c_r(\zeta) \cap i_*(\delta) = i_*(c_r(i^*\zeta) \cap \delta)$. Now we have two exact sequences on P, the first is the pull-back of the tautological sequence from Q,

$$0 \to i^*\mathcal{O}_F(-1) \to p^*E \oplus 1 \to i^*\zeta \to 0,$$

and the second is its own tautological exact sequence

$$0 \to \mathcal{O}_E(-1) \to p^*E \to \zeta' \to 0.$$

Notice that we know $i^*\mathcal{O}_F(-1) = \mathcal{O}_E(-1)$, so using the Whitney sum formula, we get that

$$c_t(\mathcal{O}_E(-1))c_t(i^*\zeta) = c_t(p^*E \oplus 1) = c_t(p^*E)c_t(1) = c_t(\zeta')c_t(\mathcal{O}_E(-1))c_t(1),$$

and since these are just equalities of non-zero polynomials, we can divide by $c_t(\mathcal{O}_E(-1))$ to get $c_t(i^*\zeta) = c_t(\zeta')c_t(1) = c_t(\zeta')$. But $c_t(\zeta')$ has no terms of degree higher than $r-1$ and thus $c_r(\zeta) = 0$, hence (i) and the proposition. □

As a preview of things to come we prove an easy version of the excess intersection formula which is a consequence of this proposition.

Corollary 1.64. *If s is the zero section of a rank r vector bundle E on X, then $s^*s_*(\alpha) = c_r(E) \cap \alpha$.*

Proof. First notice that s is indeed proper and thus allows us to define proper push-forward. This just follows since this is a closed embedding defined by modding out the higher degree terms in the definition of a vector bundle. Then let $\overline{s} = j \circ s$ be the induced section. Then we still have $q \circ \overline{s} = id$ and thus $q_* \circ \overline{s}_* = id$. Now letting $\beta = s_*(\alpha)$ in the proposition, we get $s^*s_*(\alpha) = q_*(c_r(\zeta) \cap \overline{s}_*(\alpha))$. Now from the tautological exact sequence and the Whitney sum formula, we get

$$c_t(q^*E) = c_t(q^*E \oplus 1) = c_t(\zeta)c_t(\mathcal{O}_F(-1)) = (1 + ... + c_r(\zeta)t^r)(1 + c_1(\mathcal{O}_F(-1))t),$$

upon which we see that $c_r(\zeta)c_1(\mathcal{O}_F(-1)) = 0$ since this gives term of degree $r+1$ which must be zero since q^*E is only of rank r. Since $c_r(\zeta) \neq 0$ from the proof of the proposition, we see that $c_1(\mathcal{O}_F(-1)) = 0$. Looking the degree r terms of both sides we get that $c_r(q^*E) = c_r(\zeta) + c_{r-1}(\zeta)c_1(\mathcal{O}_F(-1)) = c_r(\zeta)$. Thus we have

$$s^*s_*(\alpha) = q_*(c_r(\zeta) \cap \overline{s}_*(\alpha)) = q_*(c_r(q^*E) \cap \overline{s}_*(\alpha)) = c_r(E) \cap q_*\overline{s}_*(\alpha),$$

by the projection formula. But since $q_*\overline{s}_* = id$, we get the corollary. □

1.4 Cones and Segre Classes

1.4.1 Segre Class of a Cone

We first introduce the notion of a cone, which generalizes our geometric notion of a cone.

Definition 1.65. A cone C over a scheme X is sheaf spec of a sheaf of graded $\mathcal{O}_X$-algebras $S^\cdot$, i.e. $C = Spec(S^\cdot)$. We assume that $\mathcal{O}_X \to S^0$ is an isomorphism, S^1 is coherent, and $S^\cdot$ is generated by S^1 as an S^0-algebra. We define the cone $C \oplus 1 = S^\cdot[z]$ with n-th graded piece $S^n \oplus S^{n-1}z \oplus ... \oplus S^0z^n$. We define the projective completion of C, $P(C \oplus 1) = Proj(S^\cdot[z])$ with projection q to X. We let $\mathcal{O}(1)$ be the canonical line bundle on $P(C \oplus 1)$. The **Segre class** of the cone C, $s(C) \in A_*(X)$, is defined by

$$s(C) = q_*(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [P(C \oplus 1)])$$

.

Before we get to the details and uses of Segre classes, we should discuss a bit more about cones. Restricting to an affine open of X with coordinate ring A, we get that C is just a graded A-algebra with 0-th degree just A (really it's some homomorphic image of a polynomial ring over A since it's generated by S^1). Now an irreducible component D of C correspond to a homogeneous minimal prime ideal P of S. The coordinate ring of D is just S/P, which is a graded S^0/P^0-algebra. This is the coordinate ring of the irreducible subvariety of X corresponding to P^0. Thus each irreducible component of a cone is itself a cone over a subvariety of X. Moreover, since $S^0 = A$, we see that the union of the supporting subvarieties of the the irreducible components of C is just X itself.

Another important construction to keep in mind when dealing with cones is the projective completion mentioned above. As one usually does with the Proj construction, there is a canonical line bundle $\mathcal{O}_{C\oplus 1}(1)$ on $P(C \oplus 1)$ with a section determined by z. The zero-scheme of $z \in (S^{\cdot}[z])^1$ can easily be seen to just be $Proj(S^{\cdot})$, i.e. $P(C)$. Thus we do indeed see $P(C)$ as the hyperplane at infinity. Moreover, if we look at the complement of this zero section, i.e. $Spec(S[z]_{(z)})$, then this is isomorphic to C. Thus we see that C can be seen as an open dense subset of $P(C \oplus 1)$. From this we get that the irreducible components of C are in one-to-one correspondence with those of $P(C \oplus 1)$ and with the same multiplicites since these are local questions.

We would hope that the Segre class of a vector bundle (which is a cone with S^1 a locally free sheaf and the rest $Sym^n(S^1)$) would agree with the Segre class defined in the last chapter. Indeed it does by the following proposition:

Proposition 1.66. *(a) If E is a vector bundle on X, then $s(E) = c(E)^{-1} \cap [X]$, where $c(E)$ is the total Chern class.*

(b) Let $C_1, ..., C_t$ be the irreducible components of C with multiplicites m_i, then $s(C) = \sum_{i=1}^{t} m_i s(C_i)$.

Proof. To prove (a), we notice that $q^*[X] = [P(E \oplus 1)]$, so the definition gives $c(E \oplus 1)^{-1} \cap [X]$ from the previous chapter. But by the Whitney sum formula this is just $c(E)^{-1} \cap [X]$.

For (b), we see from the above discussion above cones that C_i is open and dense in $P(C_i \oplus 1)$ and these are the irreducible components of $P(C \oplus 1)$, so we have $[P(C \oplus 1)] = \sum m_i [P(C_i \oplus 1)]$. Using the fact that the canonical line bundle pulls back to the corresponding line bundles on each component, we get the result. □

Remark 1.67. Under the additional assumption that C is pure-dimensional and that $P(C_i) \neq \varnothing$ for each i we can use the usual pure-dimensional intersection by Cartier divisor lemma to not have to bump up to $P(C \oplus 1)$ in the definition of Segre classes. That is we get

$$s(C) = \sum_{i \geq 0} p_*(c_1(\mathcal{O}(1))^i \cap [P(C)]),$$

where $p : P(C) \to X$ is the induced projection.

1.4.2 Segre Class of a Subscheme

For a closed subscheme X of a scheme Y, we define the **normal cone** C_XY of X in Y to be $Spec(\sum_{n\geq 0}\mathcal{J}^n/\mathcal{J}^{n+1})$, where $\mathcal{J}$ is the ideal sheaf of X in Y. We define the **Segre class** of X in Y, $s(X,Y)$, to be the Segre class of the normal cone, $s(X,Y) = s(C_XY) \in A_*(X)$.

The following proposition allows us to reduce calculations of the Segre class to the case where the ambient scheme is a variety. But first we need some understanding of how blow-ups work beyond what Hartshorne mentions.

Lemma 1.68. *(a) If X is nowhere dense in Y, then $\pi : Y' = B_XY \to Y$ is a birational morphism which induces a one-to-one correspondence between the irreducible components of B_XY and those of Y which preserves multiplicities. (b) Moreover, if $X \subset Y \subset Z$ are closed imbeddings. there is a canonical imbedding of B_XY in B_XZ with the exceptional divisor of B_XZ restricting to the exceptional divisor of B_XY.*

Proof. For the first part, notice that since $E = \pi^{-1}(X)$ is a Cartier divisor on B_XY and X is nowhere dense, we get that no irreducible component B_XY can be contained in E. Indeed, if V were such a component, then it's image $\pi(V)$ would be contained in X. Now a decomposition of Y' into irreducible components Y'_i would be sent to an irreducible decomposition of Y. If $\pi(V)$ were contained in another such irreducible component, say $\pi(V')$, then $V \subset \pi^{-1}(\pi(V'))$. But since V is a maximal component on Y' we see that it must be an irreducible component of $\pi^{-1}(\pi(V'))$. But from this it follows that $\pi(V) = \pi(V')$. Thus in fact $\pi(V)$ is a maximal component of Y which is contained in X, a contradiction to the assumption. Thus $Y' - E$ meets every irreducible component of Y' and this open set is mapped isomorphicaly onto $Y - X$ byπ. The claim follows from this and the fact that these notions are defined on the local rings.

For part (b), we prove something more general which implies the result. Let $X \subset Y$ be a closed imbedding and $f : Z \to Y$ any morphism. Set $X' = f^{-1}(X)$, with $g : X' \to X$ the induced morphism. Then there is a closed imbedding $B_{X'}Z \subset B_XY \times_Y Z$ such that the exceptional divisors pull back to exceptional divisors. If $\mathcal{J}$ is the ideal sheaf of X in Y and $\mathcal{J}'$ that of X' in Z, then there is a canonical surjection from $f^*\mathcal{J}$ to $\mathcal{J}'$ of X', which induces a surjection $\oplus g^*(\mathcal{J}^n) \to \oplus \mathcal{J}'^n$ which also induces the surjection $\oplus g^*(\mathcal{J}^n/\mathcal{J}^{n+1}) \to \oplus \mathcal{J}'^n/\mathcal{J}'^{n+1}$. Applying the definitions of the blow-up construction we get the result. The chain of closed imbeddings fits into this situation. □

Lemma 1.69. *Let Y be a pure-dimensional scheme with irreducible components Y_i and multiplicities m_i. If X is a closed subscheme, set $X_i = X \cap Y_i$, then $s(X,Y) = \sum m_i s(X_i, Y_i)$.*

Proof. Let M_XY be the blow-up of $Y \times \mathbb{A}^1$ along $X \times \{0\}$. Since X is nowhere dense in $Y \times \mathbb{A}^1$, the varieties $M_{X_i}Y_i$ are the irreducible components of M_XY

with same multiplicities m_i so that

$$[M_X Y] = \sum m_i [M_{X_i} Y_i].$$

Moreover, the exceptional divisors restrict component-wise to the exceptional divisors of the components. This all follows from the previous lemma. It follows from Lemma 1.14 that upon intersecting with the exceptional divisor we get

$$[P(C_X Y \oplus 1)] = \sum m_i [P(C_{X_i} Y_i \oplus 1)].$$

Now we just cap with $c_1(\mathcal{O}(1))^i$ and sum, then push-forward to X to get the result. □

Now we get to the main result about Segre classes, namely that they are well-behaved:

Proposition 1.70. *Let $f : Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme, $g : X' \to X$ the induced morphism.*

(a) If f is proper, Y irreducible, and f maps each irreducible components of Y' onto Y, then $g_*(s(X', Y')) = \deg(Y'/Y) s(X, Y)$.

(b) If f is flat, then $g^*(s(X, Y)) = s(X', Y')$.

Proof. We define $\deg(Y'/Y) = \sum m_i \deg(Y_i'/Y)$. By the previous lemma, it suffices to prove the result when Y' is irreducible.

The set-up is as follows. Let M be the blow-up of $Y \times \mathbb{A}^1$ along the subscheme $X \times \{0\}$. From a calculation of cones, we see that the exceptional divisor is $P(C \oplus 1)$. Let M' be the blow-up of $Y' \times \mathbb{A}^1$ along $X' \times \{0\}$. From the proof of Lemma 1.68, we see that there is an induced morphism $F : M' \to M$ such that $F^* P(C \oplus 1) = P(C' \oplus 1)$ as Cartier divisors. Let G be the induced morphism from $P(C' \oplus 1) \to P(C \oplus 1)$ with projections q' (resp. q) to X' (resp. X). Then $G^* \mathcal{O}_{C' \oplus 1}(1) = \mathcal{O}_{C \oplus 1}(1)$.

For part (a), we have $F_*[M'] = d[M]$, where $d = \deg(Y'/Y)$. This follows since if we denote by $\pi' : M' \to Y' \times \mathbb{A}^1$ the blow-up projection, we have a commutative diagram

$$\begin{array}{ccc} M' & \xrightarrow{F} & M \\ {\scriptstyle \pi'}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ Y' \times \mathbb{A}^1 & \xrightarrow{f \times 1} & Y \times \mathbb{A}^1 \end{array},$$

so we get since π and π' are proper and birational,

$$d[Y \times \mathbb{A}^1] = \pi_*(F_*[M']) = (f \times 1)_*(\pi'_*[M']) = \deg(Y'/Y)[Y \times \mathbb{A}^1].$$

Thus $d = \deg(Y'/Y)$. By the projection formula, $G_*([P(C' \oplus 1)]) = d[P(C \oplus 1)]$.

So finally we get to the main calculation. For part (a) it is:

$$\begin{aligned} g_*s(X',Y') &= g_*q'_*(\sum c_1(G^*\mathcal{O}(1))^i \cap [P(C'\oplus 1)]) \\ &= q_*G'_*(\sum c_1(G^*\mathcal{O}(1))^i \cap [P(C'\oplus 1)]) \\ &= q_*(\sum c_1(\mathcal{O}(1))^i \cap d[P(C\oplus 1)]) \\ &= ds(X,Y), \end{aligned}$$

by functoriality and the projection formula.

Now for part (b) we have:

$$\begin{aligned} g^*s(X,Y) &= g^*q_*(\sum c_1(\mathcal{O}(1))^i \cap [P(C\oplus 1)]) \\ &= q'_*G^*(\sum c_1(\mathcal{O}(1))^i \cap [P(C\oplus 1)]) \\ &= q'_*(\sum c_1(G^*\mathcal{O}(1))^i \cap G^*[P(C\oplus 1)]) \\ &= s(X',Y'), \end{aligned}$$

by the flat pull-back formula for Chern classes. □

Using a blow-up, we can rephrase the Segre class in terms of more classical intersection products:

Corollary 1.71. *Let X be a proper subscheme of a variety Y. Let Y' be the blow-up along X and $X' = P(C)$ the exceptional divisor with induced projection $\eta : X' \to X$. Then*

$$s(X,Y) = \sum(-1)^{k-1}\eta_*(X'^k) = \sum \eta_*(c_1(\mathcal{O}(1))^i \cap [P(C)]).$$

Proof. From the proposition we get that $s(X,Y) = \eta_* s(X',Y')$ since Y' is birational to Y. Moreover, since X' is a Cartier divisor, its normal cone N' is a line bundle. Thus by (1.66) we see that $s(X',Y') = c(N')^{-1} \cap [X']$. But $c(N') = 1 + c_1(N')$, so $c(N')^{-1} = \sum(-1)^k c_1(N')^k$, and thus $s(X',Y') = \sum(-1)^k c_1(N')^k \cap [X']$. But from chapter two about intersecting with pseudo-divisors we see that $c^1(N')^k \cap [X'] = X'^{k+1}$. So pushing forward we get that $s(X,Y) = \sum_{k\geq 0}(-1)^k \eta_*(X'^{k+1})$ as required. For the second equality we just note that the normal bundle on X' when identified with $P(C)$ is $|mathcal O(-1)$. So $s(X',Y') = \sum c_1(\mathcal{O}(1))^k \cap [P(C)]$. Pushing forward we get the second equation. □

Remark 1.72. One immediate use of all of this theory is to calculate the degree of the morphism f. Indeed from part (a) of the proposition, with X a nonsingular point, we get that $\deg(Y'/Y) = \int_{X'} c(N')^{-1} \cap [X']$. This is useful when we know one fibre very well, but not the general fibre.

Example 1.73. *We do an example calculation of how we can calculate Segre classes by reducing to the case of an effective divisor. So let A, B, D be effective*

Cartier divisors on a surface Y. Let $A' = A + D, B' = B + D$, and let X be the scheme-theoretic intersection of A' and B'. Suppose A and B meet at a point P, non-singular on Y, and they meet transversally there. Let $f : Y' \to Y$ be the blow-up of Y at P with exceptional fibre $E = f^{-1}(P)$, $X' = f^{-1}(X)$ and induced morphism $g : X' \to X$. From a local calculation we can write $X' = f^{-1}(X) = f^*D+E$, as effective Cartier divisors. Now since this morphism is birational we get that

$$\begin{aligned} s(X,Y) &= g_*(\sum_{k\geq 0}(-1)^k X'^k \cap [X']) \\ &= g_*([X']) - g_*(X' \cdot [X']) = [D] - g_*(f^*D \cdot [f^*D] + 2f^*D[E] + E \cdot [E]) \\ &= [D] - D \cdot [D] + 2D \cdot g_*[E] + g_*(E \cdot [E]) \\ &= [D] - D \cdot [D] + [P], \end{aligned}$$

by the projection formula and the fact that $g_*[E] = 0$ since $g(E) = P$ and the fact that $E \cdot [E]$ defines the class of $-[Q]$ for any $Q \in E$ since all such points are rationally equivalent.

1.4.3 Multiplicity along a subvariety

Using all of the different approaches we have seen to calculating Segre classes, we can now define **the multiplicity of Y along X**, $e_X Y$, by the coefficient of $[X]$ in $s(X,Y)$. If we let $n = \text{codim}(X,Y)$, p, q be the projections from $P(C), P(C \oplus 1)$ to X, and Y' be the blow-up of Y along X with $X' = P(C)$ the exceptional divisor then we get from dimension considerations that

$$\begin{aligned} e_X Y[X] &= q_*(c_1(\mathcal{O}(1))^n \cap [P(C \oplus 1)]) \\ &= p_*(c_1(\mathcal{O}(1))^{n-1} \cap [P(C)]) \\ &(-1)^{n-1} p_*(X'^n). \end{aligned}$$

It turns out that this agrees with the definition of multiplicities given by Samulel which is defined by taking Hilbert-Samuel polynomials.

If $X = P$, a point, then we get the multiplicity of Y at P, $e_P Y = \int_{P(C)} c_1(\mathcal{O}(1))^{n-1} \cap [P(C)] = \deg[P(C)]$.

Similarly, for a closed subscheme X of a pure-dimensional scheme Y, we can define **the multiplicity of Y along X at V**, denoted $(e_X Y)_V$ to be the coefficient of $[V]$ in $s(X,Y)$, for any irreducible component V of X. This again is the same as Samuel's definition.

Example 1.74. *Let $f : Y' \to Y$ be a proper surjective morphism of varieties, X a closed subscheme of Y and $X' = f^{-1}(X)$. For V an irreducible component of X, assume that every irreducible component V' of $f^{-1}(V)$ has the same dimension as V. Then by (1.70a) we get that*

$$\deg(Y'/Y)(e_X Y)_V = \sum_{V'} \deg(V'/V)(e_{X'} Y')_{V'}$$

*where the sum is over the irreducible components V' of $f^{-1}(V)$. If we now let V' be a subvariety which i an irreducible compononet of $f^{-1}(V)$, $V = f(V')$, we can define $e_{V'}(f)$, the **ramification index** of f at V', to be the multiplicity of Y' along $f^{-1}(V)$ at V'. If all such V' of $f^{-1}(V)$ have the same dimension then as before we get*

$$\deg(Y'/Y)e_V Y = \sum_{V'} \deg(V'/V)e_{V'}(f).$$

In particular, if Y is smooth over an algebraically closed field, then for any $Q \in Y$ $e_Q Y = 1$ as follows from explicit calculation using any of the above definitions. Then for any Q such that $f^{-1}(Q)$ is finite we get from the above equations that $\sum_{f(P)=Q} e_P(f) = \deg(Y'/Y)$. This confirms what we know to be true for curves and shows that this sum is independent of the specific Q we choose.

We do one final example to show the equivalnce of the conventional notion of multiplicity of a point P in the special case of a Cartier divisor.

Example 1.75. *If we have a variety X of dimensionn ≥ 2, and a simple point P, take the blow-up $\pi : X' \to X$ of X at P with exceptional divisor E and induced projection $\eta : E \to P$. Let D be an effective Cartier divisor on X and $\tilde{D}$ the strict transform of D. Let m be the largest power of the maximal ideal of $\mathcal{O}_{P,X}$ which contains a local equation of D. We show that in fact $m = e_P D$, so that our definition does correspond to what one might expect. Indeed, we have $\pi^* D = \tilde{D} + mE$ as follows from a standard local equation argument. But then*

$$\begin{aligned} 0 = D \cdot \pi_*(E^{n-1}) &= \eta_*(\pi^* D \cdot E^{n-1}) \\ &= \eta_*(\tilde{D} \cdot E^{n-1}) + \eta_*(mE^n) \\ &= (-1)^{n-2} e_P(D) + (-1)^{n-1} m e_P(X). \end{aligned}$$

The first equality follows from the fact that $\pi_*(E^{n-1})$ is a class in $A_0(P)$ for dimensional reasons, so intersecting with D gives a class in degree -1, which must be zero. The second follows from the projection formula and third from the equation for $\pi^* D$. The last equation goes back to the last definition for multiplicities as intersection multiplicites of exceptional divisors. From this calculation we see that since $e_p(X) = 1$ since P is a simple point, we get that indeed $m = e_P(D)$.

1.4.4 Linear Systems

Let L be a line bundle on an n-dimensional variety X and $V \subset H^0(X, L)$ an $(r+1)$-dimensional linear system with base locus B. Let $\pi : X' \to X$ be the blow-up along B. Then from the result about resolving indeterminancies of a linear system we get a morphism $f : X' \to P(V^\vee) = \mathbb{P}^r$ extending the morphism $X - B \to P(V^\vee$ taking a point x to the hyperplane of divisors passing through x. To construct this morphism explicitly, let $\mathcal{J}$ be the ideal sheaf of B, then we get a canonical surjection $V \otimes \mathcal{O}_X \to \mathcal{J} \otimes L$ defined by sending generators for the vector space to their images in L. It happens that by the definition of B this must actually land in $\mathcal{J} \otimes L$. Thus we get a surjection $Sym^{\cdot}(V \otimes L^{-1}) \to \oplus \mathcal{J}^n$, which gives our closed imbedding of X' into $Proj(Sym(V \otimes L^{-1}) \cong Proj(Sym(V \otimes \mathcal{O}_X)) = X \times P(V^\vee)$ since we're actually taking the projective bundle of a trivial bundle. The morphism f comes from the second projection. We also see from this description that $f^*\mathcal{O}(1) = \pi^*(L) \otimes \mathcal{O}(-E)$. We can now get a useful relationship between the geometry of this morphism and the intersection theory invariants we have defined:

Proposition 1.76. *Define* $\deg_f X'$ *to be the degree of* $f_*[X']$ *as an* n*-dimensional cycle. Then* $\deg_f X' = \int_X c_1(L)^n - \int_B c(L)^n \cap s(B, X)$.

Proof. Since we can write $f^*\mathcal{O}(1) = \pi^* L \otimes \mathcal{O}(-E)$, we get

$$\begin{aligned}
\deg_f X' &= \deg(X'/f(X')) \int_{P(V^\vee)} c_1(\mathcal{O}(1))^n \cap [f(X')] \\
&= \int_{X'} c_1(f^*\mathcal{O}(1))^n = \int_{X'} (c_1(\pi^* L) - c_1(\mathcal{O}(E)))^n \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \int_X c_1(L)^{n-i} \pi_*(c_1(\mathcal{O}(E))^i \cap [X']) \\
&= \int_X c_1(L)^n - \int_B (1 + c_1(L))^n \cap s(B, X) \\
&= \int_X c_1(L)^n - \int_B c(L)^n \cap s(B, X),
\end{aligned}$$

where these last two equations follow from (1.71). □

1.5 Deformation to the Normal Cone

1.5.1 Construction of the Deformation space

Given a closed subscheme of a scheme Y with normal cone $C = C_X Y$, we construct a scheme $M = M_X Y$ together with a closed imbedding $X \times \mathbb{P}^1 \to M$ and a flat morphism $q : M \to \mathbb{P}^1$ such that the diagram

$$\begin{array}{ccc}
X \times \mathbb{P}^1 & \longrightarrow & M \\
{\scriptstyle pr}\big\downarrow & & {\scriptstyle q}\big\downarrow \\
\mathbb{P}^1 & \xrightarrow{id} & \mathbb{P}^1
\end{array},$$

commutes. Moreover, we have (1) that over $\mathbb{P}^1 - \{\infty\} = \mathbb{A}^1$, $q^{-1}(\mathbb{A}^1) = Y \times \mathbb{A}^1$ and the imbedding is just the trivial imbedding, but (2) over $\{\infty\}$, the divisor $M_\infty = q^{-1}(\infty) = P(C \oplus 1) + \tilde{Y}$, where $\tilde{Y}$ is the blow-up of Y along X. Here the imbedding of $X \cong X \times \{\infty\}$ in M_∞ is given by the zero-section imbedding of X into C followed by the canonical isomorphism of C with the open complement of $P(C)$ in $P(C \oplus 1)$. We have that $P(C \oplus 1)$ and $\tilde{Y}$ intersect in $P(C)$, which is the hyperplane at infinity in the former and the exceptional divisor in the latter. Thus the image of X in M_∞ is disjoint from $\tilde{Y}$ so taking its complement in M and letting M° be the resulting scheme, we get a family of imbeddings of $X \times \mathbb{P}^1$ into M° over $\mathbb{P}^1$ that deforms the given imbedding into the zero-section imbedding of X into C.

Proof. Construction: We start off by letting $M = B_{X\times\{\infty\}}Y \times \mathbb{P}^1$. The normal cone of $X \times \{\infty\}$ in $Y \times \mathbb{P}^1$ is just $C \oplus 1$ so the exceptional divisor is just $P(C \oplus 1)$. From the sequence of closed imbeddings

$$X \times \{\infty\} \to X \times \mathbb{P}^1 \to Y \times \mathbb{P}^1,$$

we see by (1.68b) that the blow-up of $X \times \mathbb{P}^1$ along $X \times \{\infty\}$ is imbedded as a closed subscheme of M. Since $X \times \{\infty\}$ is a Cartier divisor in $X \times \mathbb{P}^1$ (defined by the vanishing of t if s, t are the homogeneous coordinate functions on $\mathbb{P}^1$) we get by Ex II.7.11b of Hartshorne that this blow-up is just isomorphic to $X \times \mathbb{P}^1$. In other words, blowing-up a Cartier divisor doesn't do anything. This gives us our closed embedding of $X \times \mathbb{P}^1$ into M. The same argument applied to the sequence of imbeddings

$$X \times \{\infty\} \to Y \times \{\infty\} \to Y \times \mathbb{P}^1,$$

shows that $\tilde{Y}$ is imbedded in M. Since the projection $Y \times \mathbb{P}^1 \to \mathbb{P}^1$ is flat (since $\mathbb{P}^1$ is a smooth curve), we see that the composition

$$M \to Y \times \mathbb{P}^1 \to \mathbb{P}^1$$

is flat. This follows from the fact that every local ring of $\mathbb{P}^1$ is a DVR since it's a smooth curve, so the module is flat iff it's torsion free. Thus $\mathcal{O}_{Y\times\mathbb{P}^1}$ being flat over $\mathbb{P}^1$ means that it's torsion free and thus so is any ideal sheaf of it. M being defined by the direct sum of powers of an ideal sheaf imples that it's also torsion free and thus flat over $\mathbb{P}^1$, hence the claim.

Now to see statements (1) and (2) about the nature of this family of imbeddings, we first notice that (1) follows from that fact that away from $Y \times \{\infty\}$ the blow-up morphism is an isomorphism. For (2) we just need to study the local nature of M since we alreayd know that $\tilde{Y}$ and $P(C \oplus 1)$ are globally embedded in M. We investigate this local description now.

We may assume Y is affine equal to Spec A and X corresponds to the ideal I in A. Since we're interested in the behavior near ∞, we may restrict ourselves to the patch $Y \times \mathbb{A}^1$ of $Y \times \{\infty\}$ and assume that it corresponds to $Y \times \{0\}$ in this patch. Then $Y \times \mathbb{A}^1 =$ Spec $A[t]$ and the blow-up M corresponds to Proj $(\oplus((I, t)T)^n)$, that is Proj $(S^\cdot)$, where S is the graded $A[t]$-algebra with n-th component

$$S^n = I^n + I^{n-1}t + ... + At^n + At^{n+1} + ...$$

From the Proj construction, we know that M is covered by open affines Spec $S_{(x)}$, where this means the degree 0 elements in the localization S_x, as x runs over a set of generators of S^1, i.e. aT as a runs over a set of generators of I and tT. We first check one of these first kinds of affine patches, that is Spec $S_{(aT)}$ for a a generator of I. In this patch, the exceptional divisor is given by $(I,t)S_{(aT)}$ (notice that it is (I,t) without a T) and thus is a principal ideal generated by $a/1$, where now $a \in S^0 = A[t]$ is the same generator of I but without a T. This follows since for any $b \in I$, we have $b/1 = (a/1)(bT/aT)$ and $t/1 = (a/1)(tT/aT)$. So in-fact the exceptional divisor $P(C \oplus 1)$ is given by the vanishing of the equation $a/1$. Moreover, the blow-up $\tilde{Y}$ is given by setting $tT/aT = 0$ since we just blow-up the subvariety $X \times \{0\}$ inside $Y \times \{0\}$, so we just need to see the corresponding "t"-coordinate to zero. But notice now that $t/1 = (a/1)(tT/aT)$, so since we're looking for the fibre over $0 \in \mathbb{A}^1$ we get from the definition of the scheme-structure on fibres that $q^{-1}(0) = P(C \oplus 1) + \tilde{Y}$ from this description of the local coordinates for these Cartier divisors (of course translate back in your mind to the fact that our 0 was really the fibre over ∞ we wanted). Notice that the intersect of $P(C \oplus 1)$ and $\tilde{Y}$ in this patch is given by the ideal $(a/1, tT/aT)S_{(aT)}$, so modding out by this ideal eliminates the $t's$ and gives the exceptional divisor of $\tilde{Y}$.

For the other kind of patch, notice that Spec $(S_{(tT)})$ is the complement of $\tilde{Y}$ in M. Moreover, in this patch the coordinate ring of the exceptional divisor becomes $S_{(tT)}/(t/1)S_{(tT)}$, which is the same thing as modding out by t and I in each degree. i.e. Spec $\oplus\, I^n/I^{n+1} = C$. This shows that indeed we have constructed the deformation space we wanted. □

We might wonder why the heck we are interested in this deformation space. The first is that since M° is flat over $\mathbb{P}^1$, if Y is a pure n-dimensional scheme, then the class of Y in $A_n(M^\circ)$, which is $[q^{-1}(0)]$, is the same as the class of C (or $P(C \oplus 1) + \tilde{Y}$ if we're loking inside M), which is the fibre over ∞. So we have actually deformed the embedding in the Chow group. So we may perform intersection theory in C itself. Moreover, even in the best possible case that X is regularly embedded in Y with normal bundle N, then there is a vector bundle ζ on $P(N \oplus 1)$ which has a regular section vanishing precisely on X. This gives the class of X inside $P(C \oplus 1)$ as the top chern class of a vector bundle. This makes calculations a lot easier to deal with and is something not true for even a nice regular embedding.

1.5.2 Specialization to the Normal Cone

We define the specialization homomorphism $\sigma : Z_kY \to Z_kC$, which is associated to a closed subscheme X of a scheme Y with normal cone $C = C_XY$. We define σ on a k-dimensional subvariety V by setting $\sigma[V] = [C_{V\cap X}V]$ and extending linearly to all of Z_kY. Notice first that $C_{V\cap X}V$ is indeed a closed subscheme of C_XY. This follows from the proof of (1.68), which in effect shows that $C_{V\cap X}V$ is a closed subscheme of $C_XY \times_X X \cap V$. But composing with the first projection, which is itself a closed immersion as base change from the closed immersion of $X \cap V \to X$, we see that indeed $C_{V\cap X}V$ is a closed subscheme of C_XY. To see that it is purely k-dimensional, we embed $V \cap X$ in $V \times \mathbb{A}^1$ by composing with the imbedding of V in $V \times \mathbb{A}^1$ at 0. Then blow-up $V \times \mathbb{A}^1$ along $V \cap X$. Since $V \cap X$ is nowhere dense we see that this blow-up is birational to $V \times \mathbb{A}^1$ and is thus of dimension $k+1$. Since the normal cone of $V \cap X$ in $V \times \mathbb{A}^1$ is $C_{V \cap X} V \oplus 1$, so the exceptional divisor is of pure dimension k and isomorphis to $P(C_{V \cap X} V \oplus 1)$. Since $C_{V \cap X} V$ is an open dense subset of $P(C_{V \cap X} V \oplus 1)$, we get that it is indeed of pure dimension k.

As usual the big theorem is that this homomorphism passes to rational equivalence.

Proposition 1.77. *If $\alpha \sim 0$ in $Z_k(Y)$, then $\sigma(\alpha) \sim 0 \in Z_k(C)$.*

Proof. The proof of this result is kind of backward. We show that there is a homomorphism from $A_k(Y)$ to $A_k(C)$ which sends $[V]$ to $[C_{V \cap X} V]$.

Indeed, let $M^\circ = M^\circ_X Y$ be the deformation space from the previous section, i the inclusion of C in M°, and j the inclusion of $Y \times \mathbb{A}^1$ in M. Consider the commutative diagram with first row the excision exact sequence:

$$\begin{array}{ccccccc} A_{k+1}(C) & \xrightarrow{i_*} & A_{k+1}(M^\circ) & \xrightarrow{j^*} & A_{k+1}(Y \times \mathbb{A}^1) & \longrightarrow & 0 \\ & & {\scriptstyle i^*}\big\downarrow & & {\scriptstyle pr^*}\big\uparrow & & \\ & & A_k(C) & \longleftarrow & A_k(Y), & & \end{array}$$

where the map in the bottom row is one we will construct presently. Notice that since C is a principal divisor on $M^\circ - q^{-1}(0)$ defined by the vanishing of the pull-back of x_1/x_0, C has trivial normal bundle on this open set and thus by (1.43) we have $i^* i_* = c_1(N) \cap = 0$. From the fact that intersection products satisfy the flat pull back formula, and the fact C and all of the cycle classes on it are supported inside the open set $M^\circ - q^{-1}(0)$, we get that this identity holds on all of M°.

Thus we get an induced morphism from $\mathrm{Coker}(i_*) = A_{k+1}(Y \times \mathbb{A}^1)$ to $A_k(C)$. Precomposing with the flat pull-back pr^* we get the map on the bottom row of our diagram from $A_k(Y)$ to $A_k(C)$.

We just need t check that this map takes $[V]$ to $[C_{V \cap X} V]$. First we see that clearly $pr^*[V] = [V \times \mathbb{A}^1]$. From the properties of blow-ups discussed in (1.68) and the construction of the deformation space we see that $M^\circ_{V \cap X} V$ is a closed subvariety f M° and that it restricts t $V \times \mathbb{A}^1$ on the complement of C. So we may take $[M^\circ_{V \cap X} V]$ as a representative in $A_{k+1}(M^\circ$. Moreover, C intersects $M^\circ_{V \cap X} V$ in $C_{V \cap X} V$ as is clear from the construction. Thus $i^*[M^\circ_{V \cap X} V] = [C_{V \cap X} V]$ as required. □

This allows us to define the Gysin homomorphism associated to a regular imbedding $i : X \to Y$ with normal bundle N of rank d, as $i^* : A_k(Y) \to A_{k-d}(X)$, where $i^* = s_N^* \circ \sigma$.

1.6 Intersection Products

1.6.1 The Basic Construction

We start with a closed regular imbedding $i : X \to Y$ with normal bundle $N_X Y$ of rank d. Let V be a purely k-dimensional scheme with a morphism $f : V \to Y$. Let $W = f^{-1}(X)$ and form the fibre square

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ g\big\downarrow & & f\big\downarrow \\ X & \xrightarrow{i} & Y \end{array},$$

where j, g are the induced morphisms. We let $N = g^* N_X Y$ with projection $\pi : N \to W$. From the construction of the fibre square, we get that if $\mathcal{J}$ is the ideal sheaf of X in Y and $\mathcal{J}'$ is the ideal sheaf of W in V, then there is a surjection $\mathcal{J} \to \mathcal{J}'$. Taking direct sums gives a closed imbedding of $C = C_W V$ inside of N. We've seen before that since V is purely k-dimensional, so is C, and thus is defines a k-cycles $[C]$ in N. If we let s be the zero section of the bundle N, then we define the **intersection product of V by X on Y** to be $X \cdot V = s^*[C]$, where $s^* : A_k(N) \to A_{k-d}(W)$ is the Gysin homomorphism defined in chapter 3. From our equivalent definition there we see that $X \cdot V$ is the unique class on W such that $\pi^*(X \cdot V) = [C]$. We have the following proposition relating this concept to everything we've seen before:

Proposition 1.78. *(a) If we let ζ be the universal quotient bundle of rank d on $P(N \oplus 1)$ with projection q, then $X \cdot V = q_*(c_d(\zeta) \cap [P(C \oplus 1)])$*

(b)$X \cdot V = \{c(N) \cap s(W, V)\}_{k-d}$.

(c)If $d = 1$, V is a variety, and f is a closed imbedding, then $X \cdot V$ is just the intersection class defined for Cartier divisors.

Proof. From the closed imbedding $C \to N$, we get a closed imbedding of $P(C \oplus 1)$ into $P(N \oplus 1)$ and $P(C)$ into $P(N)$ all of which are compatible. Thus $P(C \oplus 1) \cap (P(N \oplus 1) - P(N)) = C$, so by (1.63)

$$\begin{aligned} X \cdot V &= s^*[C] \\ &= q_*(c_d(\zeta) \cap [P(C \oplus 1)]) \end{aligned}$$

as claimed in (a).

For (b), we consider the universal exact sequence on $P(N \oplus 1)$,

$$0 \to \mathcal{O}(-1) \to q^* N \oplus 1 \to \zeta \to 0,$$

which gives us that $c(\zeta)c(\mathcal{O}(-1)) = c(q^* N)$ from the Whitney sum formula. So we calculate using (a) that

$$\begin{aligned} X \cdot V &= q_*(c_d(\zeta) \cap [P(C \oplus 1)]) \\ &= \{q_*(c(\zeta) \cap [P(C \oplus 1)])\}_{k-d} \\ &= \{q_*(c(q^*N)c(\mathcal{O}(-1))^{-1} \cap [P(C \oplus 1)])\}_{k-d} \\ &= \{q_*(c(q^*N)s(\mathcal{O}(-1)) \cap [P(C \oplus 1)])\}_{k-d} \\ &= \{c(N) \cap q_*(s(\mathcal{O}(-1)) \cap [P(C \oplus 1)])\}_{k-d} \\ &= \{c(N) \cap s(C)\}_{k-d} \end{aligned}$$

, where the last equality just follows from tracking down the definition of $s(C)$ and using the properties of Chern and Segre classes.

For part (c) we have two cases. If $V \subset X$, then $C = W = V$, and $s^*[C] = c_1(N) \cap [W]$ from part (a) which is precisely the original intersection product by (1.43). If V is not a subset of X, then W is the pull back Cartier divisor f^*X so that $C = N$. But then $X \cdot V = \{c(N) \cap s(N)\}_{k-1} = [W] = [f^*X]$ as required. □

Definition 1.79. Let C_i be the irreducible components of the cone C from above, with multiplicities m_i so that $[C] = \sum_{i=1}^r m_i[C_i]$. Let $Z_i = \pi(C_i)$ be the support of C_i, which is a closed subvariety of W. The varieties $Z_1, ..., Z_r$ (which may have repeats) are called the **distinguished varieties** of the intersection of V by X. Let N_i be the restriction of of N to Z_i. If we let s_i be the zero section of N_i and set $\alpha_i = s_i^*[C_i]$. From the capatability of all of the morphisms involved it's clear that $\alpha_i = s^*[C_i]$. Thus $XX \cdot V = \sum m_i\alpha_i$. This is called the **canonical decomposition** of the intersection product. For each distinguished subvariety Z of W we define the equivalence of Z to be the sum of the $m_i\alpha_i$ with support Z. Note that although the actual dimension of the Z_i can be anywhere between $k-d$ and k, the classes $m_i\alpha_i$ are always supported in dimension $k-d$. If $\dim Z = k-d$, then we get that its equivalence is a multiple of $[Z]$ so we can define an associated intersection multiplicity. These last notions will be discussed in the next chapter.

Remark 1.80. Note that all of the formulas of the previous proposition hold for the distinguished varieties from the compatability of the morphisms involved.

Example 1.81. *Let $D_1 = 2A + B, D_2 = A + 2B$, where A, B are lines meeting in a point P. Let $X = D_1 \times D_2, Y = \mathbb{P}^2 \times \mathbb{P}^2, V = \mathbb{P}^2$, f the diagonal imbedding of V in Y. For the ease of calculation we can replace everything by affine opens, and we can give equations to everything. So let A be defined by $x = 0$, and B by $y = 0$. Then $X = Speck[x, y]/(x^2y) \otimes k[z, w]/(zw^2)$, $Y = Speck[x, y, z, w]$, $V = Speck[x, y]$, where the morphism f now corresponds to the k-algebra homomorphism $k[x, y, z, w] \to k[x, y]$ given by $x \mapsto x, y \mapsto y, z \mapsto x, w \mapsto y$. Then $W = Speck[x, y]/(x^2y, xy^2)$ as expected (and which follows from the properties of the tensor product. Now we calculate the cone C_WV. The ideal of W inside V is just (x^2y, xy^2), so the cone is just*

$$Spec(R) = Spec \oplus ((x^2y, xy^2)t)^n/((x^2y, xy^2)t)^{n+1}.$$

Writing out each degree as a vector space with a particularly nice basis, we see that the n-degree is spanned by all $x^a y^b t^n$, where (a,b) runs over the set of indices

$$\begin{aligned} S_n = \{(\geq 2n, n) \cup (n, \geq 2n) &\cup \{(n,2n),(n+1,2n-1),...,(2n,n)\} \\ &\cup \{(n+1,2n),...,(2n,n+1)\} \\ &\cup \{(n+1,2n+1),...,(2n+1,n+1)\}\}. \end{aligned}$$

Notice that we have a surjective homomorphism $k[x,y,u,v]/(x^2y, xy^2) \to R$ defined by $x \mapsto x, y \mapsto y, u \mapsto x^2yt^1, v \mapsto xy^2t^1$. This is surjective from a direct check. The kernel is clearly $(yu-xv)$. Thus we see that $R \cong k[x,y,u,v]/(x^2y, xy^2, yu-xv)$ and it's associated spectrum is the normal cone. From solving the equations we see that the three irreducible components are given by the ideals $(x,y),(x,u),(y,v)$, whose intersections with the degree 0 component (i.e. their images inside W) are $(x,y),(x),(y)$, respectively, i.e. the two lines A and B and the point P. We need to check the multiplicities of these irreducible components in C. If we localize R at the ideal (x,y), then u,v are invertible, so we get

$$\begin{aligned} R_{(x,y)} &= k[x,y,u,v]_{(x,y)}/(x^2y, xy^2, y - x(v/u)) \\ &= k[x,u,v]_{(x)}/(x^3(v/u), x^3(v^2/u^2)) \\ &= k[x,u,v]_{(x)}/(x^3) \end{aligned}$$

which has length 3. If we localize R at the ideal (x,u), then y,v are invertible, so we get

$$\begin{aligned} R_{(x,u)} &= k[x,y,u,v]_{(x,u)}/(x^2y, xy^2, yu - xv) \\ &= k[x,y,u,v]_{(x,u)}/(x^2, x, u - (v/y)x) \\ &= k[x,y,v]_{(x)}/(x), \end{aligned}$$

which has length 1. A similar calculation shows that $R_{(y,v)}$ has length 1. So now we can just restrict ourselves to the distinguished varieties to calculate the intersection products over them.

For the component lying over P, notice that the restriction of C to P is given by the ring $R/(x,y) = k[u,v]$ which is just a the pull back of the normal bundle N. From the definition of the intersection product, the class $[P]$ is pulled back to $[N]$ by the flat bull back π^*. So we get this component gives $3[P]$ as required from the coefficient calculation above. For the other components, we're looking at the cone $R/(x,u) = k[y,v]$ which is a rank 1 subbundle of the restriction of N to the y-axis A. Moreover, $C|_A$ is globally a component $N|_A$. The calculation for the contribution of B to the canonical decomposition is similar so we just do this calculation. Now D_i are both effective divisors in $\mathcal{O}_{\mathbb{P}^2}(3)$, so their normal bundles are both $\mathcal{O}_{\mathbb{P}^2}(-3)$. Thus the normal bundle of $D_1 \times D_2$ in $\mathbb{P}^2 \times \mathbb{P}^2$ is $\mathcal{O}(-3) \oplus \mathcal{O}(-3)$. Thus the bundle $N|_A = \mathcal{O}(-3) \oplus \mathcal{O}(-3)$ (by which I mean the total space of the corresponding locally free coherent sheaf) and $C|_A$ is a component of this bundle. If we take the function u which cuts out $C|_A$ on this *affine patch of $A = \mathbb{P}^1$ then on the other patch this gets equated with u/x_1^3, so we have a rational function u/x_1^3 with $(u/x_1^3) = [C] - 3[Q]$, where Q is any point on $\mathbb{P}^1$. So now we're looking for a class α on A such that $\pi^*\alpha = [C] = 3[Q]$. Clearly $[3Q]$ will do the job, so the component in the canonical decomposition is precisely this (the coefficient was 1 from our calculation above). Thus we finally get that*

$$X \cdot V = \alpha + \beta + 3[P],$$

where α (resp. β) is a zero cycle of degree 3 on A (resp. B).

Example 1.82. *It is not always the case that the intersection product is commutative as the following example demonstrates. Let $Y = \mathbb{P}^2$, X_1 the curve $xy = 0$, X_2 the curve $x = 0$ and P the point $x = y = 0$. The intersection product of X_2 by X_1 has only X_2 as a distinguished subvariety, while the intersection product of X_1 by X_2 has both X_2 and P distinguished. This can be shown and the intersection product calculated using the same methods as before.*

Example 1.83. *We show that (1.78) gives us another description of the intersection product. Indeed from its proof we see that $\{c(N) \cap s(W, V)\}_i = q_*(c_{k-i}(\zeta) \cap [P(C \oplus 1)])$. If $i < k - d$, then $k - i > d$ which is greater than the rank of ζ. Thus $c_{k-i}(\zeta) = 0$ as does $\{c(N) \cap s(W, V)\}_i = 0$ for $i < k - d$. Thus $c(N) \cap s(W, V) = X \cdot V +$ higher terms, so $X \cdot V$ can be described as the lowest degree term in the class $c(N) \cap s(W, V)$.*

1.6.2 Refined Gysin Homomorphisms

Definition 1.84. Given a regular imbedding of codimension d, $i : X \to Y$, let $f : Y' \to Y$ be a morphism, and form the fibre square

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ X & \xrightarrow{i} & Y \end{array} .$$

We define the **refined Gysin homomorphism** $i^! : A_k(Y') \to A_{k-d}(X')$ as follows. On cycles we define

$$i^!(\sum n_i[V_i]) = \sum n_i X \cdot V_i.$$

From this definition it's not entirely obvious that the homomorphism passes to rational equivalence. So we notice that $C' = C_{X'}Y'$ is a closed subcone of $N = g^*N_XY$, and that $i^!$ is the composite

$$Z_k(Y') \to Z_k(C') \to A_k(N) \to A_{k-d}(X'),$$

where the first map is the specialization map σ from the deformation to the normal cone, the second map is just inclusion, and the last map is the Gysin morphism s^* for the zero-section of X' in N. But we know that σ passes to rational equivalence so $i^!$ does as well. If $f = id_Y$ then we just denote the refined Gysin morphism by i^*.

Theorem 1.85. *Consider a two-tiered fibre diagram*

$$\begin{array}{ccc} X'' & \xrightarrow{i''} & Y'' \\ {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow \\ X' & \xrightarrow{i'} & Y' \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ X & \xrightarrow{i} & Y \end{array},$$

with i a regular imbedding of codimension d.

(a) (Push-forward) If p is proper then $i^! p_* \alpha = q_* i^! \alpha \in A_{k-d}(X')$.

(b) (Pull-back) If p is flat of relative dimension n, then $i^! p^* \alpha = q^* i^! \alpha \in A_{k+n-d}(X'')$.

(c) (Compatibility) If i' is also a regular imbedding of codimension d then $i^! \alpha = i'^! \alpha \in A_{k-d}(X'')$.

Proof. The proofs of (a) and (b) are very similar so we just present the proof of (a). Clearly we may prove the result on classes of subvarieties, so let $\alpha = [V']$ and $V = p(V')$. Then

$$\begin{aligned} i^! p_*[V'] &= \deg(V'/V)\{c(N) \cap s(X' \cap V, V)\}_{k-d} \\ &= \{c(N) \cap q_*(s(X'' \cap V', V'))\}_{k-d} \\ &= q_*\{c(q^*N) \cap s(X'' \cap V', V')\}_{k-d} \\ &= q_* i^! [V']. \end{aligned}$$

For part (c), it is enough to show that $g^* N_X Y = N_{X'} Y'$ since then all of the morphisms in the above decomposition of $i^!$ are the same. To show this, let $\mathcal{J}$ and $\mathcal{J}'$ be the ideal sheaves of X and X' in Y and Y', respectively. Then we have a canonical surjection $g^*\mathcal{J} \to \mathcal{J}'$ from the fact that this is a fibre square. This induceds a surjection $g^*(\mathcal{J}/\mathcal{J}^2) \to \mathcal{J}'/\mathcal{J}'^2$. Since these are both locally free sheaves of the same rank this must be an isomorphism as claimed. □

Remark 1.86. Notice that from the flat pull-back property we may calculate the components of an intersection product in a connected component by replacing the ambient scheme by an open subscheme.

1.6.3 Excess Intersection Formula

Consider again a 2-tiered fibre diagram

$$\begin{array}{ccc} X'' & \xrightarrow{i''} & Y'' \\ {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow \\ X' & \xrightarrow{i'} & Y' \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ X & \xrightarrow{i} & Y \end{array},$$

with i (resp. i') regular imbeddings of codimension d (resp. d') and normal bundle N (resp. N'). From the usual game the canonical surjections on ideal sheaves induces a canonical closed imbedding of N' in g^*N. We set $E = g^*N/N'$, a vector bundle of rank $e = d - d'$ called the **excess normal bundle**. We have the following theorem:

Theorem 1.87. *(Excess Intersection Formula) For* $\alpha \in A_k(Y'')$,

$$i^!\alpha = c_e(q^*E) \cap i'^!\alpha \in A_{k-d}(X'').$$

Proof. Let $Q' = P(q^*N' \oplus 1), Q = P(q^*g^*N \oplus 1)$ with universail quotient bundles ζ' and ζ on Q' and Q, respectively. There is a canonical imbedding of Q' in Q such that the tautological line bundle on Q restricts to the same line bundle on Q'. From applying the Whitney sum formula to the corresponding tautological exact sequences we see that $c(r^*q^*g^*N) = c(\mathcal{O}_{Q'}(-1))c(\zeta|_{Q'}), c(r^*q^*N') = c(\mathcal{O}_{Q'}(-1))c(\zeta')$, where $r^* : Q' \to X''$ is the projection. Dividing the first equation by the second and cancelling the common term, we get that

$$c(\zeta|_{Q'}) = c(r^*q^*E)c(\zeta'),$$

and if we take top degree terms then we see that $c_d(\zeta|_{Q'}) = c_e(r^*q^*E)c_{d'}(\zeta')$. Now it clearly suffices to prove the theorem on subvarieties V of Y'', so let $\alpha = [V]$ and $P = P(C_{V\cap X''}V \oplus 1)$. Then we have from (1.78)

$$\begin{aligned} i^!(\alpha) &= r_*(c_d(\zeta|_{Q'}) \cap [P]) \\ &= r_*(c_e(r^*q^*E)c_{d'}(\zeta') \cap [P]) \\ &= c_e(q^*E) \cap r_*(c_{d'}(\zeta') \cap [P]), \text{by the projection formula,} \\ &= c_e(q^*E) \cap i'^!(\alpha). \end{aligned}$$

□

Corollary 1.88. *Suppose we have a fibre square*

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ X & \xrightarrow{i} & Y \end{array},$$

with i a regular imbedding of codimension d and normal bundle N and i' an isomorphism. Then

$$i^!(\alpha) = c_d(g^*N) \cap \alpha.$$

In partcular,

$$i^* i_*(\alpha) = c_d(N) \cap \alpha.$$

Proof. The first assertion is clear from the theorem. For the second, known as the **self-intersection formula**, we consider the fibre diagram

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ {\scriptstyle id}\downarrow & & {\scriptstyle i}\downarrow \\ X & \xrightarrow{i} & Y, \\ {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow \\ X & \xrightarrow{i} & Y \end{array}$$

and then by (1.85)

$$i^* i_* \alpha = i^!(\alpha) = c_d(N) \cap \alpha,$$

where this last equality is from the first statement of of this corollary. □

We have the following proposition which says that intersection products commute with Chern classes.

Proposition 1.89.

1.7 Intersection Multiplicites

1.7.1 Proper Intersections

With notation as in the previous chapter we have the following lemma:

Proposition 1.90. *(a) Every irreducible component of W is distinquished.*
(b) For any distinguished variety Z, $k - d \leq \dim Z \leq k$.

Proof. To prove part (a), notice that we've seen already the irreducible components of a cone are themselves cones over their supports, and the union of the supports is all of W. Thus removing the redundancies we get the irreducible components of W from amongst the Z_i.

For part (b), notice that since C_i is an irreducible subvariety of N which projects onto Z_i, we get that $C_i \subset N_i$, where N_i is the restriction of N to Z_i. Thus

$$\dim Z_i \leq \dim C_i \leq \dim N_i = \dim Z_i + d,$$

from which the claim follows since we know C_i is k-dimensional. □

If $\dim Z_i = k - d$, the inclusion $C_i \subset N_i$ must be an isomorphism since these are both irreducible varieties of dimension k. Thus the equivalence of Z_i is precisely $m_i Z_i$.

Definition 1.91. An irreducible component of W is called a **proper component** of the intersection of V by X if $\dim Z = k - d$. The **intersection multiplicity** of Z in $X \cdot V$, denoted $i(Z, X \cdot V; Y)$, is the coefficient of Z in the class $X \cdot V$. From the description before the definition, we see that for such a Z the cone above Z is just N_Z, the restriction of N to Z, so we see that $i(Z, X \cdot V; Y)$ is the multiplicity of N_Z in the cycle $[C]$.

Let $A = \mathcal{O}_{Z,V}$ be the local ring of an irreducible subvariety Z inside of W and J the ideal corresponding to W. Then A/J has finite length when Z is an irreducible component of W since then this is the zero-dimensional local ring of Z in W.

Proposition 1.92. *Suppose Z is a proper component of W. Then*

(a)$1 \leq i(Z, X \cdot V; Y) \leq l(A/J)$.

(b) If J is generated by a regular sequence of length d, then $i(Z, X \cdot V; Y) = l(A/J)$. Moreover, if A is CM then the local equations for X in Y give a regular sequence generating J and hence the equality holds.

Proof. As above N_Z is an irreducible component of N. Since N is a vector bundle, the coefficient of N_Z in N is the same as the coefficient of Z in W, i.e. $l(A/J)$. This just follows for example from the first lemma about flat pull-backs. Now C is a closed subscheme of N so we must have that the coefficient of N_Z in $[C]$ no larger than its coefficient in $[N]$. Indeed the coefficient of N_Z in $[N]$ is given by $l_{\mathcal{O}_{N_Z,N}}(\mathcal{O}_{N_Z,N})$, while the coefficient of N_Z in $[C]$ is $l_{\mathcal{O}_{N_Z,C}}(\mathcal{O}_{N_Z,C})$. But since C is a closed subscheme of N defined by an ideal, say I, we see that $\mathcal{O}_{N_Z,C} = \mathcal{O}_{N_Z,N}/I$. Thus the coefficient of N_Z in $[C]$ is in fact $l_{\mathcal{O}_{N_Z,N}}(\mathcal{O}_{N_Z,N}/I) \leq l_{\mathcal{O}_{N_Z,N}}(\mathcal{O}_{N_Z,N})$ from the additivity of length onf exact sequences. The result now follows from the last sentence in (1.80).

For part (b) we notice that we may replace V by an open subscheme which meets Z since the intersection multiplicity is local in nature. Thus in this case we may assume that $Z = W$ is imbedded in V regularly with codimension d. Then C is a sub-bundle of N of the same rank, d, so $C = N$ and thus the coefficients must of N_Z in $[C]$ and $[N]$ must agree. For the last part of (b) we notice that since Z has the "correct" codimension, J is generated by a regular sequence from standard commutative algebra about CM rings. □

Remark 1.93. From (1.78b) and dimension considerations we see that $i(Z, X \cdot V; Y) = (e_W V)_Z$ since the effect of capping with the Chern class is none when there are components of the right dimension. From our previous comments about the multiplicities of a scheme along a subscheme at a subvariety we see that our definition agrees with Samuel's.

Example 1.94. *We give an example where strict inequality occurs in (1.81a). Let $Y = \mathbb{A}^4$, X the plane $(x - z, y - w)$ and V the union of two planes meeting at a point given by $(x, y) \cap (z, w) = xz, xw, yz, yw)$. Then indeed V is purely 2-dimensional with irreducible components each of the individual planes with multiplicity 1. By linearity, we can compute the intersection number for $P = (0, 0, 0, 0)$ on each component separately. But the clearly X intersects each* *component exactly once at the origin. Thus* $i(P, X \cdot V; Y) = 2$. *But if we calculate the length of the corresponding ring, we see that*

$$k[x, y, z, w]/(xz, xw, yz, yw, x - z, y - w) \cong k[x, y]/(x^2, xy, y^2),$$

which has length 3. So indeed we get strict inequality.

Example 1.95. *We discuss another example in which strict inequality occurs. Let* $Y = \mathbb{A}^4$, $X = V(x, w)$, *and* $V \subset \mathbb{A}^4$ *the image of the finite morphism* $\phi : \mathbb{A}^2 \to \mathbb{A}^4$ *given by* $\phi(s, t) = (s^4, s^3t, st^3, t^4)$. *We notice that the intersection* $X \cap V$ *has induced reduced scheme structure given by precisely the point* $P = (0, 0, 0, 0)$. *This follows from just solving the equations defining the map. Thus we see that* P *is the proper component of the intersection. A long calculation shows that* $I(V) = (xw - yz, x^2z - y^3, yw^2 - z^3, y^2w - z^2x)$. *Now* $[W] = [X \cap V]$, *which as we have said has underlying space the point* P. *But we check the multiplicity of the corresponding local ring. The coordinate ring of the intersection is*

$$\begin{aligned} k[x, y, z, w]/(xw - yz, x^2z - y^3, yw^2 - z^3, &y^2w - z^2x, x, w) \\ &\cong k[x, y, z, w]/(yz, y^3, z^3, x, w) \\ &\cong k[y, z]/(yz, y^3, z^3), \end{aligned}$$

which has length 5. Thus we see that $[X \cap V] = 5[P]$ *since* $5 = l(A/J)$ *in this case. But we'll see presently that this is not the intersection multiplicity.*

We first notice the degree of the morphism ϕ *is 4 since for example the preimage of the origin* $P \in \mathbb{A}^4$ *is the origin* $Q \in \mathbb{A}^2$ *with multiplicity 4 since we're solving the equations* $s^4 = 0$ *and* $t^4 = 0$. *Thus* $\phi_*[\mathbb{A}^2] = 4[V]$. *Applying the theorem about proper push-forwards and refined Gysin morphisms to the fibre diagram,*

$$\begin{array}{ccc} V(s^4, t^4) & \longrightarrow & \mathbb{A}^2 \\ {\scriptstyle\psi}\downarrow & & {\scriptstyle\phi}\downarrow \\ X \cap V & \longrightarrow & V\,, \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ i\ } & Y \end{array}$$

we see that $\psi_*(i^![\mathbb{A}^2]) = i^!\phi_*[\mathbb{A}^2] = 4X \cdot V$. *Of course by part (b) of (1.81), since* $\mathbb{A}^2$ *is regular and thus CM we get that*

$$i(Q, X \cdot \mathbb{A}^2; Y) = l(k[s, t]/(s^4, t^4)) = 16.$$

From the compatibility of intersection products we get that $16 = i(Q, X \cdot \mathbb{A}^2; Y) = 4i(P, X \cdot V; Y)$, *so* $i(P, X \cdot VV; Y) = 4 \neq 5$.

Example 1.96. *A final example which shows what can go wrong. We don't calculate intersection multiplicities since we show in this case it doesn't make* *sense to. Without the ambient space being regular, we sometimes wind up cutting down the dimension by too much. For example, if* $Y = V(xw - yz) \subset \mathbb{A}^4$*,* $X = V(x,y)$*,* $V = V(z,w)$*, then* $X \cap V = V(xw - yz, x, y, z, w) = V(x,y,z,w)$*, i.e. just the origin. But we'd expect the irreducible components to be of dimension at least* $\dim V - codim(X,Y) = 2 - 1 = 1$.

From the various theorems about the refined Gysin morphisms, we get the following properties about intersection multiplicities:

Commutativity If $V \to Y$ is also a regular imbedding, then $i(Z, X \cdot V; Y) = i(Z, V \cdot X; Y)$.

Associativity If $i : X \to Y$ factors into $i' : X \to X'$, $j : X' \to Y$, with both regular imbeddings, then let $W'_1, ..., W'_r$ be the irreducible components of $f^{-1}(X')$ which contain Z. If Z is a proper component of W, then Z is a proper component of the intersection of each W'_h by X on X', and each W'_h is a proper component of the intersection of V by X' on Y, so we get that

$$i(Z, X \cdot V; Y) = \sum_{h=1}^{r} i(Z, X \cdot W'_h; X') \cdot i(W'_h, X' \cdot V; Y).$$

Projection formula If $g : V' \to V$ is a proper surjective morphism of k-dimensional varieties, and $Z_1, ..., Z_r$ are the irreducible components of $g^{-1}(Z)$, then if each Z_j and Z has dimension $k - d$, then

$$\deg(V'/V) \cdot i(Z, X \cdot V; Y) = \sum_{j=1}^{r} \deg(Z_j/Z) \cdot i(Z_j, X \cdot V'; Y).$$

2 Complex Algebraic Surfaces

2.1 Generalities and important theorems

The first important and useful theorem is the Riemann-Roch Theorem for surfaces:

Theorem 2.1 (Riemann-Roch)**.** *For all* $L \in Pic\ X$*,*

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}(L^2 - L \cdot K).$$

We have another Riemann-Roch-like theorem which connects the geometry of algebraic surfaces and the underlying topology:

Theorem 2.2 (Noether's formula)**.** $12\chi(\mathcal{O}_X) = K^2 + \chi_{top}(X)$

Finally we have a result which relates the intersection theory of the surface with genus of an irreducible curve lying on it.

Proposition 2.3 (Genus formula)**.** *If we define, as usual, the arithmetic genus of a curve as* $p_a(C) = h^1(C, \mathcal{O}_C)$*, then*

$$p_a(C) = 1 + \frac{1}{2}(C^2 + C \cdot K).$$

An easy proof of this fact for irreducible curves follows from the ideal sheaf exact sequence, but using RR it follows for all effective curves.

One of the deepest and most interesting results in the theory of surfaces (and one which can be generalized to higher dimensional birational geometry) is Castelnuovo's contractibility criterion:

Theorem 2.4. *Let X be a surface and E a rational (-1)-curve. Then E can be contracted to a point by a birational morphism to a smooth surface.*

We now introduce the Hodge theoretic invariants that govern so much of the geometry and topology of our complex surfaces. We define $p_g(X) = h^2(X, \mathcal{O}_X)$, and $q(X) = h^1(X, \mathcal{O}_X)$ so that $\chi(\mathcal{O}_X) = 1 - q + p_g$. From Hodge theory we have that $2b_1(X) = q(X)$. From Poicare duality we have $\chi_{top}(X) = 2 - 2b_1(X) + b_2(X)$. Moreover, an important invariant is the plurigenus, $P_n(X) = h^0(X, nK)$.

These invariants come up in distinguishing the various Kodaira dimension 0 surfaces. As a first step in that direction we have another theorem of Castelnuovo which determines when a surface is rational:

Theorem 2.5. *Let X be a surface with $q = P_2 = 0$, then X is rational.*

We also have a characterization of ruled surfaces given by Enriques:

Theorem 2.6. *A surface is ruled iff $P_n = 0$ for all $n \geq 1$ iff it contains a non-exceptional curve C with $K \cdot C < 0$.*

The following result will also be useful when we discuss Enriques surfaces and K3 surfaces, two of the main classes of Kodaira dimension 0 surfaces.

Lemma 2.7. *Let X be minimal with $\kappa = 0$. Then*
(a) $K^2 = 0$;
(b) $\chi(\mathcal{O}_X) \geq 0$;
(c) if $P_n = P_m = 1$ then $P_d = 1$ for $d = (n, m)$.

Proof. First note that we have Beauville's "useful remark" that if C is an irreducible curve such that $C^2 \geq 0$, then $C \cdot D \geq 0$ for any effective divisor D. Then we have that since the Kodaira dimension is 0 $P_n = 1$ for some integer n. But if C is an irreducible curve such that $K \cdot C < 0$, then since X is not ruled (because $\kappa \geq 0$), we have there are no non-exceptional curves with $K \cdot C < 0$ (a consequence of Enriques' theorem), and thus $C^2 \geq 0$. But then by the useful remark $P_n = 0$ for all n, a contradiction. Thus $K \cdot C \geq 0$ for any irreducible curve. Summing we get that this is true for all effective divisors as well. But then since $|mK|$ contains an effective divisor D for some m, we must have $D \cdot K \geq 0$, but this implies that $K^2 \geq 0$. Suppose $K^2 > 0$. Then by RR we have that $h^0(nK) + h^0((1-n)K) \geq \chi(\mathcal{O}_X) + \left(\binom{n}{2}\right) K^2$. It's easy to see from the above argument that $h^0((1-n)K) = 0$ for $n \geq 2$. Thus $P_n \to \infty$ with n, contradicting the Kodaira dimension. For the part about the Euler characteristic, we have from part (a) and Noehter's formula that $12\chi(\mathcal{O}_X) = \chi_{top}(X) = 2 - 2b_1(X) + 2b_2(X) = 2 - 4q + 2b_2$, which is equivalent to $8\chi(\mathcal{O}_X) = 2 - 4q + 2b_2 - 4\chi(\mathcal{O}_X) = 2 - 4q + 2b_2 - 4(1 - q + p_g) = -2 - 4p_g + 2b_2$.

But since $p_g \leq 1$, $8\chi(\mathcal{O}_X) \geq -6$, and because the Euler characteristic is an integer we must have $\chi(\mathcal{O}_X) \geq 0$. Suppose $D \in |nK|$ and $E \in |mK|$. Write $n = n'd, m = m'd$ with $(n', m') = 1$. Then $m'D$ and $n'E$ belong to the same zero dimensional linear system $|\frac{nm}{d}K|$ (the dimensionality follows from the fact that $\kappa = 0$ and we know it's nonempty for n and m) and thus are equal. So we have an effective divisor Δ such that $D = n'\Delta$ and $E = m'\Delta$. Now consider the class $\epsilon \in \text{Pic } X$ given by $\Delta - dK$. Then $n'\epsilon \equiv m'\epsilon \equiv 0$. By the properties of the order of an element, we get that since $(m', n') = 1$, $\epsilon \equiv 0$. Thus $P_d = 1$. □

2.2 Surface Singularities

We will see that a very interesting type of singularity, called rational singularities, can be described in three equivalent ways. We explore the first method by means of resolutions of these singularities.

2.2.1 Minimal resolutions

We define a resolution of the singularity $x \in X$, for X a surface, to be a birational morphism $\pi : Y \to X$, where Y is smooth and $x \in \pi(Y)$. A minimal one is one that cannot be factored through another resolution. If the point $x \in X$ is normal then since $\mathcal{O}_{X,x}$ is regular in codim 1 we get that locally around x there are no other singularities, so we may take an affine open subset U such that π is an isomorphism over $U\backslash\{x\}$. The fibre over x is connected by Zariski's Main Theorem. By Castelnuovo's Theorem, a resolution is minimal iff Y has no (-1)-self-intersection curves. In the direction of a classification of rational surface singularities, we have the following proposition:

Proposition 2.8. *Let $E_1, ..., E_n$ be some irreducible components of the exceptional curve E of a resolution. Then the matrix $(E_i \cdot E_j)_{i,j}$ is negative definite.*

Proof. This proof is due to Mumford. We replace X, as above, with the open affine subset U such that π is an isomorphism above $U\backslash\{x\}$. Choose a $u \in \mathcal{O}(U)$ which vanishes at x, and let $\Phi = \pi^*(u)$. By Krull's Hauptidealsatz, we get that every associated prime ideal of Φ has codimension 1, and thus $div(\Phi) = \sum n_i E_i + Z$, where Z is the proper transform of some one-dimensional subscheme passing through x (i.e. the one defined by the vanishing of u). Thus Z intersects at least one component E_k. We now do a straightforward calculation to finish off. We first show that for any $a_1, ..., a_n \in \mathbb{Q}$,

$$(\sum a_i n_i E_i)^2 = -\sum_{i<j}(a_i - a_j)^2 n_i n_j (E_i \cdot E_j) - \sum a_i^2 n_i E_i \cdot Z.$$

For notational simplicity, set $D_i = n_i E_i$, then we get

$$(\sum a_i D_i)^2 = \sum_i a_i (D_i \cdot \sum_j a_j D_j) = \sum_i a_i D_i(-a_i div(\Phi) + \sum_j a_j D_j),$$

since multiplying with a principal divisor is zero. Of course, writing this out gives

$$\sum_i a_i D_i \cdot \sum_j ((a_j - a_i)D_j - a_i Z) = -\sum_{i<j} (a_i - a_j)^2 D_i \cdot D_j - \sum_i a_i^2 D_i \cdot Z.$$

Of course, now we're done since the first term is at most zero whereas each term in the second summand is at most zero while at least one is strictly negative. Thus we're done. □

Note that this result also implies that the E_i are rationally and numerically independent.

We define an **exceptional cycle** as a positive integral combination of irreducible components of the exceptional curve. The **genus** of a normal singularity is defined to be $\dim_k (R^1\pi_*\mathcal{O}_Y)_x$. This is independent of the resolution because of the result on the factorization of a birational morphism of smooth surfaces as a sequence of blow-ups. Note that the coherent sheaf $R^1\pi_*\mathcal{O}_Y$ is supported at the point x since π is an isomorphism away from x. Thus $R^1\pi$ is really just H^1 of the fibers. Since all fibers but the one at x are just single points, H^1 of the structure sheaf just vanishes. Moreover, it follows that if $\pi(Y)$ is affine and π is an isomorphism away from x, then this is just $\dim_k H^1(Y, \mathcal{O}_Y)$ by the definition of R^1. We say that a singularity is **rational** if its genus is 0. We have the following characterization of rational singularities:

Proposition 2.9. *The following are equivalent for a normal singularity:*

(i) x is a rational singularity;

(ii) $H^1(Z, \mathcal{O}_Z) = 0$ for every exceptional cycle Z;

(iii) $p_a(Z) := 1 + \frac{1}{2} Z \cdot (Z + K_Y) \leq 0$ for every exceptional cycle Z;

(iv) the homomorphism from Pic(Z) to $\mathbb{Z}^r$ defined by $\mathcal{L} \mapsto (..., \deg(\mathcal{L} \otimes \mathcal{O}_{E_i}), ...)$, is bijective, for every exceptional cycle;

(v) the canonical map $H^1(X, \mathcal{O}_X) \to H^1(Y, \mathcal{O}_Y)$ is bijective.

Proof. We prove everything but the equivalence with (iv). We noted above that $R^1\pi_*\mathcal{O}_Y$ is concentrated at x. So by the theorem on formal functions (III, 11.1 in Hartshorne) we have that the completion of the stalk at x is isomorphic to $\varprojlim H^1(Z_{(r)}, \mathcal{O}_{Z_{(r)}})$, where $(r) = (r_1, ..., r_n) \to \infty$ and $Z_{(r)} = \sum r_i E_i$. Also note that this stalk is zero iff the completion is zero since completion is a fully faithful functor. Notice that really we can take the base space for all of these cohomologies to just by the reduced scheme structure on the fibre $E = \cup E_i$ since this does not effect the cohomology. Moreover, since the dimension of this space is 1, the short exact sequence of sheaves $0 \to \mathcal{K} \to \mathcal{O}_{Z_{(r)}} \to \mathcal{O}_{Z_{(s)}} \to 0$, where $r_i \geq s_i$ for each i, gives a surjection $H^1(E, \mathcal{O}_{Z_{(r)}}) \to H^1(E, \mathcal{O}_{Z_{(s)}})$ from the vanishing of H^2 given by Grothendieck's Vanishing Theorem applied to $\mathcal{K}$. Thus $H^1(Z, \mathcal{O}_Z) = 0$ for all positive exceptional cycles Z iff the singularity is rational. This gives the equivalence of (i) and (ii). The equivalence with (iii) follows from (ii) and the fact that $p_a(Z) = 1 - \chi(Z) = 1 - (h^0(Z) - h^1(Z)) = 1 - h^0(Z) + h^1(Z) \leq h^1(Z)$ since $h^0(Z)$ is equal to the number of connected components of Z which is at least one. To show the equivalence of all of the above with (v) we consider the exact sequence given by the Leray spectral sequence. Namely we consider

$$0 \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_Y) \to \Gamma(R^1\pi_*\mathcal{O}_Y) \to H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_Y) \to 0.$$

So clearly the bijectivity of this map is equivalent to the definition of rational singularity. Note also that these conditions are equivalent to the H^2's being in bijection. Thus these singularities are those for which resolving them does not change the arithmetic genus. □

There is a specific exceptional divisor that is very important for us. We define a **fundamental cycle** $Z = \sum m_i E_i$ to be an exceptional cycle with all positive coefficients such that (a) $Z \cdot E_i \leq 0$ for every i, and (b) Z is minimal among all exceptional cycles satisfying (a). This definition is useful because of the following proposition:

Proposition 2.10. *A fundamental cycle Z exists and satisfies $p_a(Z) \geq 0$. A singularity is rational iff $p_a(Z) = 0$. In this case the multiplicity of the point $x \in X$ (i.e. the length of $\mathcal{O}_{X,x}$) is equal to $-Z \cdot Z$. Moreover the dimension of the Zariski Tangent space at this point is $-Z \cdot Z + 1$.*

Proof. We won't prove that equivalence for rational singularities, though this isn't difficult and is contained in Artin's paper. Likewise with the multipllicity calculation. To see the existence of the fundamental cycle, note that if $Z_1 = \sum r_i^1 E_i$ and $Z_2 = \sum r_i^2 E_i$ both satisfy $Z \cdot E_i \leq 0$ for each i, then so does $Z = \sum r_i E_i$, where $r_i = \inf(r_i^1, r_i^2)$ and Z is positive. This follows since if, for example, $r_j^1 \leq r_j^2$, then

$$(Z \cdot E_j) = r_j^1 E_j^2 + \sum_{i \neq j} r_i (E_i \cdot E_j) \leq r_j^1 E_j^2 + \sum_{i \neq j} r_i^1 (E_i \cdot E_j) = (Z^1 \cdot E_j) \leq 0.$$

It follows that since each Z^i is greater than E itself, so must Z be. It follows from this that their exists a unique smallest one, the fundamental cycle. To see the relevant inequality with the arithmetic genus we suppose Y to be a positive cycle not greater or equal to Z. Then $Y \cdot E_i > 0$ for some i. By Ex V.1.3(c) in Hartshorne, we get $p_a(Y + E_i) = p_a(Y) + p_a(E_i) + (Y \cdot E_i) - 1$. Since $p_a(E_i) \geq 0$, $p_a(Y) \leq p_a(Y + E_i)$. If in fact we have $0 < Y < Z$, then the multiplicity of this E_i in Y is less than that of E_i in Z, so $Y + E_i \leq Z$. Since, for example, $p_a(E_1) \geq 0$, we can build up using these facts to get that also $p_a(Z) \geq 0$ □

A rational singularity of multiplicity 2 is called a **rational double point**. The key fact in the classification of rational double points is the following:

Proposition 2.11. *Let $\pi : Y \to X$ be a minimal resolution of a normal singularity and $E = \cup E_i$ the exceptional curve with its irreducible components. Then x is a rational double point iff $E_i \cong \mathbb{P}^1$ and $E_i^2 = -2$ for each i.*

Proof. First suppose that x is a rational double point. If we let $Z = \sum m_i E_i$ be the fundamental cycle, then by the proposition above we get that $Z^2 = -2$ and $p_a(Z) = 0$. This automatically means $Z \cdot K_Y = 0$ from the formula for the arithmetic genus (which follows from the general RR for surfaces or exercise in Hartshorne), which then imples that $\sum m_i E_i \cdot K_Y = 0$. Now from Proposition 1.2 we know that $H^1(E_i, \mathcal{O}_{E_i}) = 0$, and by a standard result this means $E_i \cong \mathbb{P}^1$. Of course since π is a minimal resolution, $E_i^2 < -1$, so since $0 = H^1(E_i, \mathcal{O}_{E_i}) = p_a(E_i) = 1 + \frac{1}{2}(E_i^2 + E_i \cdot K_Y)$, we get $E_i \cdot K_Y \geq 0$, and thus from the equation $Z \cdot K_Y = 0$ we must have $E_i \cdot K_Y = 0$ and thus $E_i^2 = -2$. Conversely, if $E_i \cong \mathbb{P}^1$ and $E_i^2 = -2$ for all i, then $E_i \cdot K_Y = 0$. Thus $Z \cdot K_Y = 0$ for every exceptional cycle $Z = \sum m_i E_i$. By Proposition 1.1 $Z^2 < 0$ and since it's even (just from the definition and the fact that $E_i^2 = -2$) we must have $Z^2 \leq -2$. Thus $p_a(Z) \leq 0$, so by Proposition 1.2 we have x is a rational singularity. If we take the fundamental cycle, then $p_a(Z) = 0$, from which we get that $Z^2 = -2$, so x is a double point. □

This leads us directly into the following classification of ratoinal double points, also known as the ADE classification.

Theorem 2.12. *Let E be the exceptional curve of a minimal resolution of a rational double point. Then $E_i \cdot E_j \geq 0$ for $i \neq j$ and $E_i \cap E_j \cap E_k = \varnothing$ for $i \neq j \neq k$. This allows us to assign a Dynkin diagram to the associated Cartan matrix, which are classified as in the theory of Lie algebras. That is since we have only simply laced diagrams, we get singularities of types A_n, D_n, E_6, E_7, E_8. Moreover, the fundamental cycle is a maximal root (i.e. subtracting any postive root from it results in a positive root) for the integral lattice generated by the classes of the E_i and can be explicitly given as (order the components bottom-to-top and left-to-right):*
(i) $Z = E_1 + ... + E_n$ for type A_n;
(ii)$Z = E_1 + E_2 + 2E_3 + ... + 2E_{n-1} + E_n$ for type D_n;
(iii)$Z = 2E_1 + E_2 + 2E_3 + 3E_4 + 2E_5 + E_6$ for type E_6;
(iv)$Z = 2E_1 + 2E_2 + 3E_3 + 4E_4 + 3E_5 + 2E_6 + E_7$ for type E_7;
(v)$Z = 3E_1 + 2E_2 + 4E_3 + 6E_4 + 5E_5 + 4E_6 + 3E_7 + 2E_8$ for type E_8.

Proof. The associated intersection matrix is negative definite with -2 on the diagonal and non-negative entries else-where. It's indecomposable since E is connected. This is enough to plug into the classification of theory of Cartan matrices to get that adding to this matrix $2I_n$ gives the incidence matrix of the mentioned Dynkin diagrams. The triple intersections being empty follows from those diagrams. □

2.2.2 Local Analytic equations

Furthermore, we can describe the local (analytic) coordinates of these classes of singularities. We present the result over $\mathbb{C}$, but a similar result exists for all characteristics, but with many more classes.

Proposition 2.13. *Any rational double point is analytically isomorphic to one of the following singularities and conversely:*
(i) A_n*:* $z^{n+1} + xy = 0$ *for* $n \geq 1$*;*
(ii) D_n*:* $z^2 + x(y^2 + x^n) = 0$ *for* $n \geq 4$*;*
(iii) E_6*:* $z^2 + x^3 + y^4 = 0$*;*
(iv) E_7*:* $z^2 + xy^3 + x^3 = 0$*;*
(v) E_8*:* $z^2 + x^3 + y^5$*.*

Proof. First note that by simply going through the explicit calculations for the minimal resolution by repeatedly blowing up singular points we get a minimal resolution of these surface singularities that give the arrangement described in the first part. So a singularity described by these equations is rational. To see the other direction we must use the work of Tjurina (*On the tautness of rationally contractible curves on a surface*), which says that the singularity is determined by the incidence matrix of the exceptional curves. □

It is interesting to note that all rational double points can be realized as the singularities of a double cover of a smooth surface. Important to distinguish also are the singluarities of type A_1, which are called **ordinary double points**. Note that these have an irreducible exceptional rational curve with self-intersection -2.

A possibly important fact is the following:

Proposition 2.14. *A rational singularity is Gorenstein iff it is a rational double point. Moreover, in this case* $\pi^*(\omega_X) \cong \omega_Y$*.*

Proof. From the calculation of the dimension of the tangent space, we see that locally the singularity is embeddable in codimension 1 in $\mathbb{C}^3$ iff it's a double point. Indeed the result is clear in one direction. For the opposite direction lift linearly independent generators of $\mathfrak{m}/\mathfrak{m}^2$ to $\mathfrak{m}$. These generate a subring of dimension 3 which is isomorphic to a polynomial ring in these variables, so this gives an inclusion locally of at least the local ring into $\mathbb{C}^3$ which becomes an isomorphism on tangent spaces, i.e. an embedding. This becomes an isomorphism upon completion so the singularities are indeed isomorphic, but now the singularity is one on a surface in $\mathbb{C}^3$. Moreover, this embedding singularity is a hyperplane singualrity, i.e.it's codimension 1. We now show then that a singularity is Gorenstein iff it embeds in comdimension 1, which will finish the first claim. It is "apparently well known" that hypersurfaces, and indeed complete intersections, are Gorenstein. If X is Gorenstein, then the canonical divisor K_X corresponds to a line bundle ω_X which is just sheaf of rational 2-forms. Because this resolution is minimal $E_i \cdot K_Y \geq 0$. Indeed this follows from the fact that $E_i^2 < -1$ from the Hodge Index Theorem and minimality. So since $0 \leq p_a(E_i) = 1 + \frac{1}{2}(E_i^2 + E_i \cdot K)$, we must have $E_i \cdot K_Y \geq 0$. Moreover, since X is Gorenstein and π is an isomorphism away from x, we must have that $K := K_Y - \pi^* K_X = \sum a_i E_i$. Now, writing $K = A - B$, where A and B are linear combinations of the E_i without common components and both are nonnegative, we get that $A \cdot B \geq 0$, and $A^2 \leq 0$, so $(A - B) \cdot A \leq 0$. On the other hand, $(A - B) \cdot A = K \cdot A = (K_Y - \pi^* K_X) \cdot A = K_Y \cdot B \geq 0$ by above. This forces $A = 0$, so $K \leq 0$. Now if $K < 0$, then $-K > 0$ and since x is a rational singularity we have that $p_a(-K) \leq 0$ by Proposition 1.9. On the other hand, $p_a(-K) = 1 + \frac{1}{2}(K^2 - K \cdot K_Y) = 1$, which is a contradiction. Thus $K = 0$ and $K_Y = \pi^* K_X$, which proves the last claim. It also shows that $K_Y \cdot E_i = 0$ for all i. The converse direction in the proof of Proposition 1.11 shows that x must then be a rational double point which we saw was embeddable in codimension 1. □

I should mention that this last fact, that the minimal resolution of the singularity is crepant, in fact characterizes rational double points, as we pretty much showed.

2.2.3 Quotient Singularities

We now present the final face of rational double points, namely that they are obtained as quotient singularities of $\mathbb{C}^2$ by a finite subgroup of $SL_2(\mathbb{C})$.

2.3 Del Pezzo Surfaces

While not so important from the point of view of classification of complex algebraic surfaces, these surfaces exhibit beautiful geometry even though they are non-minimal rational surfaces.

Definition 2.15. A **Del Pezzo surface** X is a smooth projective surface with ample anticanonical bundle.

Some include rationality as part of the definition of a Del Pezzo surface, but this is infact unneccessary. Indeed, since $-K_X$ is ample, mK_X couldn't be effective for any $m \geq 1$, so $h^0(mK_X) = 0$. It follows from Kodaira vanishing that $h^1(mK_X) = 0$ for all $m \in \mathbb{Z}$ (for negative m is follows from Serre's cohomological criterion for ampleness). Furthermore, by Serre duality $h^2(mK_X) = h^0((1 - m)K_X)$. Thus $q = h^1(\mathcal{O}_X) = 0$ and $P_2 = h^0(mK_X) = 0$, so by Theorem 1.5 we see that X is rational.

We call $d = K_X^2$ the **degree** of the Del Pezzo surface X. We will derive a classification of Del Pezzo surfaces and we start with the following general lemma about rational surfaces.

Lemma 2.16. *The Picard group of a rational surface X is a finitely generated free abelian group whose rank satisfies*

$$rk\ Pic\ X + K_X^2 = 10.$$

Proof. Since X is rational, there is a birational map $X \dashrightarrow \mathbb{P}^2$, so by the theorem on factorization of birational maps we know there exists a third surface X' and maps $\pi_1 : X' \to X$ and $\pi_2 : X' \to \mathbb{P}^2$ which are compositions of blow-ups at closed points. Now if X is a rational surface and $\pi : X' \to X$ is the blow-up at one closed point, then Pic $X' =$ Pic $X \oplus \mathbb{Z}E$, where E is the exceptional curve and $K_{X'} = \pi^* K_X + E$. Thus rk Pic X' =rk Pic $X + 1$ and $K_{X'}^2 = K_X^2 - 1$, so the two sums agree. Thus the lemma is true for X if and only if it's true for X'. Working up and down the above factorization we find that we just need to check the theorem for $\mathbb{P}^2$, but here it's clear since $K_{\mathbb{P}^2} = -3H$ and Pic $\mathbb{P}^2 \cong \mathbb{Z}$. □

Proposition 2.17. *Let X be a Del Pezzo surface of degree d. Then $1 \leq d \leq 9$. Furthermore, every irreducible curve with negative slef-intersection is exceptional. If none exist then either $d = 9$ and $X \cong \mathbb{P}^2$ or $d = 8$ and $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. The first assertion follows directly from the lemma since both K_X^2 and rk Pic X are at least 1 (the former because $-K_X$ is ample). For the second assertion, suppose C is a curve with negative self-intersection. Then because $-K_X$ is ample we have $-K_X \cdot C > 0$ and $0 \leq p_a(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_X)$, which forces $C^2 = -1$ and $p_a(C) = 0$. Thus C is a (-1)-curve and thus exceptional by Castelnuovo's Theorem. For the third assertion, we must have X minimal since it has no exceptional curves and it furthermore has no curves with negative self-intersection number. Now from Proposition IV.1 in Beauville we see that for $n > 0$ $\mathbb{F}_n$ has a unique irreducible curve B with negative self-intersection $-n$. Thus it must be one of only two minimal rational surfaces which satisfy this, $\mathbb{P}^2$ or $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. This completes the proof of the theorem. □

Ths big theorem about Del Pezzo surfaces is the following classification theorem:

Theorem 2.18. *Let X be a Del Pezzo surface of degree d. Then*
(i) If $d = 9$, then X is isomorphic to $\mathbb{P}^2$;
(ii) if $d = 8$, then X is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown-up at one point.
(iii) If $1 \leq d \leq 7$ then X is isomorphic to $\mathbb{P}^2$ blown-up at $9 - d$ closed points, no three of which lie on a line and no six of which lie on a conic.

Proof. We already know the minimal Del Pezzo surfaces, so we may assume X is not minimal. Take one of X's minimal models, i.e. a birational morphism $f : X \to W$, with W a minimal rational surface, a priori either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$. If it's $\mathbb{F}_n$ with $n > 0$, then to be minimal $n > 1$ so we have a unique irreducible curve with negative self-intersection, say C. Then $C^2 \leq -2$. Now if f were just one blow-up, then $f^*C = \tilde{C} + mE$, where $\tilde{C}$ is the proper transform of C under the birational morphism f, and m is the multiplicity of C through the center of the blow-up and E is the exceptional curve. Thus $(f^*C)^2 = \tilde{C}^2 + 2m\tilde{C} \cdot E - m^2 = \tilde{C}^2 + 2m^2 - m^2 = \tilde{C}^2 + m^2$, so since $C^2 = (f^*C)^2 = \tilde{C}^2 + m^2$ we have at each stage of the blow-up $\tilde{C}^2$ is reduced, so then we have an irreducible curve $\tilde{C}$ on X with $\tilde{C}^2 \leq -2$. But this contradicts part (ii) of our proposition. Thus $W \cong \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In the second case, let $x \in W$ be a point at which f isn't defined. Then from the factorization theorem on birational morphisms of surfaces we have that f factors through the blow-up at x, say through $\pi : W' \to \mathbb{P}^1 \times \mathbb{P}^1$. But we can blow-down on W' the two proper images of the lines on W through x. This gives us a birational morphism from

W' to $\mathbb{P}^2$, and thus a birational morphism from X to $\mathbb{P}^2$, which we also denote by f.

Since each blow-up in the factorization of f increases the rank of the Picard group by one, we get that we need exactly rk Pic $X-$rk Pic $\mathbb{P}^2 = 10 - d - 1 = 9 - d = r$ blow-ups to factor f. Let $x_1, ..., x_s$ be the centers of these blow-ups on $\mathbb{P}^2$. Clearly $s \leq r$, but we show that in fact $s = r$. If not then one of the blow-ups in f's factorization would have its center on an exceptional curve of a previous blow-up. Let E be this exceptional curve and π be the blow-up of a point on it. Then $\tilde{E}$ under this map has self-intersection -2 (1 less than the self intersection of E, i.e. -1). This would only descrease on X as we saw above. This contradicts the proposition. Thus $s = r$. Now suppose that three of the points lie on a line L. Then since $L^2 = 1$ we have that the blow up at these three points has proper image of L with self-intersection -2. Blowing up further would only decrease this number, a contradiction to the proposition. The same argument with a conic through six points gives the same contradiction. □

The truth is that the converse of this result is true for $d \geq 3$. But we develop some preliminaries first.

Proposition 2.19. *Let $\mathfrak{d}$ be the linear system of plane cubics with assigned base points $P_1, ..., P_r$, and assume that no 4 or the P_i are collinear, and no 7 of them lie on a conic. If $r \leq 7$, then $\mathfrak{d}$ has no unassigned base points (that is base points of the induced linear system on the blow-up at these points).*

Proof. Clearly it is sufficient to consider the case $r = 7$. For this it suffices to show that for any point Q not equal to one of the P_i, there is a cubic curve containing $P_1, ..., P_7$ but not Q. There are 3 cases:

Case 1 Suppose that Q is on a line L with some three points, say P_1, P_2, P_3. Then since the remaining points are not collinear we may choose three which are not collinear, say P_4, P_5, P_6. Then the unique conic through P_1, P_2, P_4, P_5, P_6 together with the line through P_3 and P_7 forms a cubic C through the P_i but not through Q. Indeed if Q were on the conic, then L would be contained in it, so it's the union of two lines. The second line must thus contain P_4, P_5, P_6 since no 4 points are collinear. But this contradicts our assumption. If Q were on the line between P_3 and P_7, then this would have to be the same line as L, so P_1, P_2, P_3, P_7 are collinear, again a contradiction.

The other cases are similar and invole considering whether Q is on a conic containing 6 of the points or not. □

Corollary 2.20. *With the same hypotheses as above, we get that if $r \leq 8$, then* $\dim \mathfrak{d} = 9 - r$, *and if $r = 8$ then* $\dim \mathfrak{d} = 1$ *and almost every curve in the linear system is irreducible.*

Proof. For $r \leq 7$, there are no unassigned base points, so at each step, the dimension drops by one. The cubics in $\mathbb{P}^2$ form a system of dimension 9. This proves (a). For (b), we note that no 4 points collinear and no 7 on a conic, there are only finitely many ways of passing three lines or one line and one irreducible conic through eight points. □

Corollary 2.21. *Given 8 points $P_1, ..., P_8$ in the plane, no 4 collinear, and no 7 lying on a conic, there is a uniquely determined point P_9 such that every cubic passing through $P_1, ..., P_8$ passes through P_9.*

Proof. From the previous corollary, we can find two distinct irreducible cubics through $P_1, ..., P_8$. By Bezout's theorem these intersect in 9 points, the first eight of which are the P_i's. Since $\dim \mathfrak{d} = 1$, any other cubic is a linear combination of these (otherwise this would be a 1-dimensional subsystem of the same dimension). Thus it must also pass through P_9. □

Lemma 2.22. *Let $\mathfrak{d}$ be the linear system of plane cubic curves with base points $P_1, ..., P_r$, and assume that no 3 of the P_i are collinear, and no 6 of them lie on a conic. If $r \leq 6$, then the corresponding linear system $\mathfrak{d}'$ on the blow-up of $\mathbb{P}^2$ at $P_1, ..., P_r$ is very ample.*

Proof. We need to check that this linear system separates points and tangent vectors. To separate points just means that there are no unassigned base points. This is true by Proposition 1.19. To separate tangent vectors means that there are no unassigned base points in the linear system subsystem passing through P as well, for any point P. But the hypotheses of the lemma guarantee that $P_1, ..., P_r, P$ satisfy the hypotheses of Proposition 1.19. So we're done. □

We can finally prove the converse to our classification theorem:

Theorem 2.23. *Let X' be the surface obtained by blowing up the r points from the lemma. Then X' is embedded by $\mathfrak{d}$ into $\mathbb{P}^{9-r}$ as asurface of degree 9-r whose canonical bundle satsifies $\omega_{X'} \cong \mathcal{O}_{X'}(-1)$.*

Proof. By the lemma the system is very ample. The fact about dimension follows from earlier work. If L is a line in $\mathbb{P}^2$, then $\mathfrak{d}' = |\pi^* 3L - E_1 - ... - E_r|$, so for any $D' \in \mathfrak{d}'$, $D'^2 = 9 - r$. So the degree of X' in $\mathbb{P}^{9-r}$ is $9-r$. Finally, since the canonical divisor on $\mathbb{P}^2$ is $-3L$ we get that $K_{X'} = \pi^*(-3L) + E_1 + ... + E_r = -D'$. Thus $\omega_{X'} \cong \mathcal{O}_{X'}(-1)$. □

We thus see that these are indeed Del Pezzo surfaces as defined above.

2.3.1 Cubic surfaces in $\mathbb{P}^3$

A very famous and classical result is easily obtained from our results on Del Pezzo surfaces. Since cubic surfaces X in $\mathbb{P}^3$ satisfy $\omega_X \cong \mathcal{O}_X(-1)$ we see that all nonsingular cubic surfaces are Del Pezzo and by numerical restrictions we see that every nonsingular cubic surface is the blow-up of $\mathbb{P}^2$ at 6 points no three collinear and no six contained in a conic. From this description we easily obtain the classical description of 27 lines on a smooth cubic surface in $\mathbb{P}^3$.

Theorem 2.24. *Every smooth cubic surface X contains exactly 27 lines. Each has self-intersection -1, and they are the only irreducible curves with negative self-intersection. They are the six exceptional curves E_i, the strict transforms F_{ij} of the fifteen lines between the points P_i and P_j, and the strict transforms G_j of the six conics containing all but one of the points.*

Proof. We first notice that if L is a line on X, then L has degree 1, and has arithmetic genus 0. Since $K_X = -H$, we get that $1 = H \cdot L = -K_X \cdot L$ and thus $0 = p_a(L) = 1 + \frac{1}{2}(L^2 + L \cdot K_X)$ implies that $L^2 = -1$. Conversely, if C is an irreducible curve with $C^2 < 0$, then since $p_a(C) \geq 0$, the genus formula imples that $p_a(C) = 0, C^2 = -1, \deg C = -K_X \cdot C = 1$, so C is a line.

This makes our work purely combinatorial. First notice that the degree of all of the curves we've listed is 1. So we just need to check that there are no others. Suppose C is such a curve. If C is not one of the E_i then we make write it as $C \sim aL - \sum b_i E_i$ for $a > 0, b_i \geq 0$, where L is the pull-back of a line from $\mathbb{P}^2$. Then $\deg C = 3a - \sum b_i = 1$ and $C^2 = a^2 - \sum b_i^2 = -1$. Now from the Cauchy-Schwartz inequality, $(\sum b_i)^2 \leq 6 \sum b_i^2$, so we find that $3a^2 - 6a - 5 \leq 0$. Solving we find that $a = 1$ or 2. If $a = 1$, then $b_i = b_j = 1$ for some i, j and o for the rest, i.e. F_{ij}. If $a = 2$, then $b_i = 1$ except for $b_j = 0$, i.e. G_j. □

2.4 Enriques Surfaces

2.4.1 Double Covers

Before getting to specific examples, we must of course mention the strong correspondence between Enriques surfaces and K3 surfaces. This connected to the notion of a double cover. A morphism $f : X \to Y$ is a **double cover** if it's finite and of degree 2. If we suppose X is CM and Y is smooth, then by Theorem 18.16.b. in Eisenbud we have that $f_* \mathcal{O}_X = \mathcal{A}$ is a locally free coherent sheaf of $\mathcal{O}_Y$-algebras of rank 2. Moreover, we have by Ex. II.5.17 in Hartshorne that $X \cong \mathbf{Spec}\ \mathcal{A}$. If we take an affine cover $\{U_i = \text{Spec } A_i\}$ that trivializes $\mathcal{A}$, then $\mathcal{A}(U_i) \cong A_i[T]/(T^2 + a_i T + b_i)$. Clearly 1 and t_i generate $\mathcal{A}(U_i)$ as a free A_i-module of rank 2. Comparing on different patches we find that $t_i = g_{ij} t_j + c_{ij}$ for some $g_{ij} \in \mathcal{O}_Y(U_i \cap U_j)^*$ and $c_{ij} \in \mathcal{O}_Y(U_i \cap U_j)$. Thus $\mathcal{A}$ is isomorphic to an extension $0 \to \mathcal{O}_Y \to \mathcal{A} \to \mathcal{L} \to 0$, where $\mathcal{L}$ is an invertible sheaf with transition functions $\{g_{ij}\}$. The functions $\{c_{ij}\}$ define a cocyle in $H^1(Y, \mathcal{L}^{-1})$, which determines the class of this extension. We can also determine how the coefficients patch together: $a_i = g_{ij}(a_j - 2c_{ij})$ and $b_i = g_{ij}^2 b_j - g_{ij} a_j c_{ij} - c_{ij}^2$. In particular, if we replace t_i by $t_i + \frac{1}{2} a_i$, then we may assume $a_i = 0$ and then $c_{ij} = 0$, so the extension splits. Thus $\mathcal{A} \cong \mathcal{O}_Y \oplus \mathcal{L} \cong A_i[T]/(T^2 + b_i)$, where the first expression is as modules and the second as algebras. From the simplified equation for the $\{b_i\}$ we get that they patch together to define a section b of $\mathcal{L}^{-2}$. And conversely, a line bundle $\mathcal{L}$ and a section of $\mathcal{L}^{-2}$ define a double cover. Now, from the local description above it's clear that X is ramified above the curve given by $div(b)$. In particular if b doesn't vanish, then the cover is unramified.

Since in characteristic zero double covers are really hypersurfaces in Z,the total space of the line bundle corresponding to $\mathcal{L}$, we can calculate their canonical bundle. Indeed, we get that $p^* \mathcal{L} \cong \mathcal{O}_Z$ and $\omega_Z \cong p^*(\omega_Y \otimes \mathcal{L})$, where $p : Z \to Y$ is the projection of the line bundle. This much just follows from results in Hartshorne. From adjunction (which we can use since X is a complete intersection in this case), we get that

$$\omega_X \cong (\omega_Z \otimes p^*\mathcal{L}^{-2})|_X \cong p^*(\omega_Y \otimes \mathcal{L}^{-1})|_X.$$

Since the inclusion of X into Z commutes with projection we in fact get that $\omega_X \cong f^*(\omega_Y \otimes \mathcal{L}^{-1})$.

We have the following topological relationship for branched covers of smooth surfaces:

Proposition 2.25. *$\chi_{top}(X) = 2\chi_{top}(Y) - \chi_{top}(B)$, where B is the ramification curve.*

Proof. We have the canonical extension $0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathcal{L} \to 0$. Taking Euler characteristics, we get $\chi(f_*\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \chi(\mathcal{L})$, and since f is finite we have $\chi(f_*\mathcal{O}_X) = \chi(\mathcal{O}_X)$. By RR we have $\chi(\mathcal{L}) = \chi(\mathcal{O}_Y) + \frac{1}{2}(\mathcal{L}^2 + \mathcal{L} \cdot K_Y)$. From Noether's formula we get that

$$\chi_{top}(X) = 12\chi(\mathcal{O}_X) - K_X^2 = 12(\chi(\mathcal{O}_Y) + \chi(\mathcal{L})) - (f^*(K_Y - \mathcal{L}))^2.$$

The rest just follows from putting all of this together and using the genus formula. □

2.4.2 Definitions and properties

Definition 2.26. An **Enriques surface** X is a minimal smooth projective surface with Kodaira dimension $\kappa = 0$, and $p_g = q = 0$.

Proposition 2.27. *If X is an Enriques surface, then $2K \sim 0$. Moreover, from Noether's formula $b_2(X) = 10$.*

Proof. We have that $P_2 \neq 0$ since by Castelnuovo's Rationality Theorem we would have that X is rational and thus ruled. But a surface is ruled iff $P_n = 0$ for all $n \geq 1$. Now for any $\kappa = 0$ surface there is some n such that $P_n = 1$. Thus we have $P_2 \neq 0$. By RR, we have that

$$h^0(-2K) + h^0(3K) \geq 1 + \frac{1}{2}((-2K)^2 - (-2K) \cdot K) = 1.$$

Now if $P_3 = 1$, then so does P_1, which is not the case. Thus $h^0(-2K) \geq 1$. This implies $2K \equiv 0$. By Noether's formula $12 = 12(1 - 0 + 0) = 12\chi(\mathcal{O}_X) = \chi_{top}(X) = 2 - 4 \cdot 0 + b_2$, so $b_2(X) = 10$. □

We can now describe important groups associated with these Enriques surfaces.

Proposition 2.28. *For an Enriques surface X, $NS(X) \cong H^2(X, \mathbb{Z})$ and $Num(X) \cong H^2(X, \mathbb{Z})/torsion \cong \mathbb{Z}^{10}$, an integral lattice equipped with a non-degenerate bilinear form which induces a perfect pairing.*

Proof. From the exponential exact sequence,

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$

we get a long exact sequence on cohomology,

$$H^1(X, \mathcal{O}_X) \to \text{Pic } X \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X).$$

But since $q = p_g = 0$ for an Enriques surface, we get Pic $X \to H^2(X, \mathbb{Z})$, the first chern class homomorphism, is an isomorphism. From the theory of Picard schemes, the subgroup of divisors algebraically equivalent to zero is an abelian variety of dimension $b_1(X) = 2q = 0$, i.e. a point. Thus Pic $X = NS(X)$. Moreover, from this isomorphism we get that the torsion subgroups of $NS(X)$ and $H^2(X, \mathbb{Z})$ are isomorphic. Modding out by these we get that $Num(X) \cong H^2(X, \mathbb{Z})/torsion \cong \mathbb{Z}^{10}$, and the perfect pairing from the intersection product comes from Poincare duality. In other words, the integral lattice is unimodular. □

An important consequence of the above is that the cohomology of an Enriques surface is algebraic. Indeed since c_1 : Pic $X \to H^2(X, \mathbb{Z})$ is an isomorphism, and we know that $H^2(X, \mathbb{C})$ is concentrated in $H^{1,1}$, which is where the cohomology of algebraic curves (or divisors) comes from, we see that all of the cohomology comes from algebraic curves. This is in contrast with K3 surfaces as we'll see. Moreover, unlike the variance in the Picard lattice of K3 surfaces, we get the following result about the Picard lattice of an Enriques surface, also known as the **Enriques lattice**.

Proposition 2.29. *Let X be an Enriques surface. Then $Num(X)$ is an even unimodular hyperbolic lattice of rank 10 isomorphic to $U \oplus E_8$, where U is the hyperbolic plane and E_8 is the standard exceptional rank 8 lattice.*

Proof. We've already seen that $Num(X)$ is a unimodular lattice of rank 10. Since K_X is numerically trivial, we see from the genus formula that $C^2 = 2p_a(C) - 2 \in 2\mathbb{Z}$ for any irreducible curve C. Thus it's even. Moreover, by the Hodge index theorem, $Num(X)$ has signature $(1, 9)$, and thus is hyperbolic. By Theorem 5 of Ch. 5 in [Ser], we get the result. □

Thus all Enriques surfaces have the same Picard lattice, which alreayd suggests that they form one family.

Like with all surfaces with numerically trivial canonical class, the RR formula takes on a simpler form for Enriques surfaces X: $\chi(\mathcal{O}_X(D)) = \frac{1}{2}D^2 + 1$. Moreover, since the arithmetic genus is given by $p_a(D) = \frac{1}{2}D^2 + 1$, we get $\chi(\mathcal{O}_X(D)) = p_a(D)$. Also, if D is a strictly positive divisor, then the last term in the definition of $\chi(\mathcal{O}_X(D))$, $h^0(X, \mathcal{O}_X(K_X - D)) = h^2(X, \mathcal{O}_X(D))$, vanishes upon considering the exact sequence

$$0 \to \mathcal{O}_X(K_X - D) \to \mathcal{O}_X(K_X) \to \mathcal{O}_D(K_X) \to 0,$$

and the fact that ω_X has no global sections since it's torsion. We have the following additional vanishing theorem for Enriques surfaces:

Proposition 2.30. *Let C be an irreducible curve on an Enriques surface X with $C^2 > 0$. Then $H^1(X, \mathcal{O}_X(C)) = H^1(X, \mathcal{O}_X(K_X + C)) = 0$.*

Proof. This follows from both the Kodaira vanishing theorem and the Nakai-Moishezon criterion for ampleness. Indeed if we define $C' := K_X + C$, then C' is numerically equivalent to C □

Although we don't get to K3 surfaces proper until the next section, we would be remiss if we didn't mention the easiest source of examples of Enriques surfaces and the strong connection they have with K3's. This is summed up in the following proposition:

Proposition 2.31. *There is a one-to-one correspondence between Enriques surfaces and K3 surfaces with a fixed-point free involution.*

Proof. The line bundle ω_Y is 2-torsion and thus defines an unramified double cover $\pi : X \to Y$. This X is then smooth, and if we notice that $\pi^*\omega_Y \cong (\mathcal{O}_Y \oplus \omega_Y) \otimes \omega_Y \cong \mathcal{O}_Y \oplus \omega_Y \cong \mathcal{O}_X$, then since the covering is unramified we get $K_X \equiv \pi^* K_Y \equiv 0$. Moreover, from standard facts about free actions of finite groups in Mumford's book, we get $\chi(\mathcal{O}_X) = 2\chi(\mathcal{O}_Y) = 2$. From all of this we see that X is K3. Conversely, given a fixed-point free involution, we get $2K_Y \equiv \pi_* \pi^* K_X \equiv 0$. Since $\chi(\mathcal{O}_Y) = 1$, we get enough to conclude Y is Enriques from the classification of $\kappa = 0$ surfaces. □

2.5 K3 Surfaces

2.5.1 Definition and first properties

Definition 2.32. A **K3 surface** X is a minimal smooth projective surface with Kodaira dimension $\kappa = 0$, and $p_g = 1, q = 0$.

Proposition 2.33. *If X is a K3 surface then $K \sim 0$ and $b_2(X) = 22$.*

Proof. Since $p_g = 1$, we have $\chi(\mathcal{O}_X) = 2$, so by RR $h^0(-K) + h^0(2K) \geq 2$. Since $P_n \leq 1$ for any n, we get that $h^0(-K) = 1$, so K is trivial. By Noether's formula, $24 = 2\chi(\mathcal{O}_X) = \chi_{top}(X) = 2 - 4 \cdot 0 + b_2(X)$, so $b_2(X) = 22$. □

2.5.2 Cohomology and the Picard lattice

Similar to our approach to the cohomology of Enriques surfaces, we can move somewhat in that direction. But the lack of vanishing of the arithmetic genus prevents us from determining the Picard number of a K3 surface. We will see later that this is where the importance of polarizations come in.

Proposition 2.34. *For X a K3 surface, Pic $X = NS(X) = Num(X) \cong \mathbb{Z}^\rho$, where $\rho \leq 22$.*

Proof. Again we look at the exponential exact sequence and the associated long exact sequence on cohomology,

$$H^1(X, \mathcal{O}_X) \to \text{Pic } X \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X).$$

Since $q = 0$ we know that at least the first chern class homomorphism is injective. Furthermore, since $b_1(X) = 2q = 0$, we get again Pic $X = NS(X)$ as in the Enriques case. Moreover, we claim that its torsions subgroup is trivial. Indeed suppose that $L \in$ Pic X is a torsion element. Then by RR $h^0(L) + h^0(L^{-1}) \geq 2+\frac{1}{2}(L^2 - L\cdot K) = 2+\frac{1}{2}L^2$. Now since L is torsion it has 0 self-intersection. Thus either L or L^{-1} has a non-zero section. But a torsion element of the Picard group that has non-zero global sections must be trivial. So in fact Pic $X \cong Num(X)$ injects into $H^2(X, \mathbb{Z})$. Since $Num(X)$ is a free abelian group sitting inside a finitely generated abelian group of rank 22, we see that all three of these groups are isomorphic to $\mathbb{Z}^\rho$, with $\rho \leq 22$. □

2.5.3 Examples, Polarizations, and Moduli

While we don't go into a formal discussion of moduli spaces, we explore some examples and the necessity of polarizations. But first let us discuss normal K3 surfaces, that is ones whose embedding into projective space corresponds to a complete linear system. We have the following result:

Proposition 2.35. *Let $X \subset \mathbb{P}^n$ be a normal K3 surface. Then it has degree $2n - 2$.*

Proof. Let H be a hyperplane section for this embedding. Then by definition $h^0(X, \mathcal{O}(H)) = n + 1$. By Serre duality, $h^2(X, \mathcal{O}(H)) = h^0(X, \mathcal{O}(-H)) = 0$, since $K_X \equiv 0$ and H is effective. Moreover, by Kodaira vanishing and Serre duality $h^1(X, \mathcal{O}(H)) = h^1(X, \mathcal{O}(-H)) = 0$. So by RR,

$$n + 1 = \chi(\mathcal{O}_X) + \frac{1}{2}H^2 = 2 + \frac{1}{2}H^2.$$

Since for a very ample divisor H, H^2 is the degree of the corresponding emebedding, we see that $\deg X = H^2 = 2n - 2$. Also note that from the genus formula this is also the genus of a generic section. □

The choice of such a curve (or more generally an ample divisor) is called a **polarization**. We explore some examples of normal K3's (with the corresponding polarization) before delving into more general polarizations.

The easiest examples of normal K3's are hypersurfaces and complete intersections. Amongst these the easiest are quartic hypersurfaces in $\mathbb{P}^3$. By the Lefschetz hyperplane theorem, a smooth quartic X in $\mathbb{P}^3$ has irregularity $q(X) = q(\mathbb{P}^3) = 0$. This also follows from Ex. III.5.5(c) in Hartshorne. Then by the adjunction formula or Ex II.8.4(e) in Hartshorne, we have $K_X \equiv -4H + 4H \equiv 0$, so X is a K3 surface. Now the family of smooth quartics in $\mathbb{P}^3$ is a quasi-projective variety $U \subset \mathbb{P}^34$ of dimension 35-1=34. Since an automorphism of such a "polarized" K3 surface must preserve the polarization, it must come from a projective automorphism. Since dim $PGL_4 = 15$, we get that the moduli space of such polarized K3 surfaces is of dimension 34-15=19, and is the GIT quotient $U//PGL_4$.

The next example to consider is the sextic K3 X in $\mathbb{P}^4$ We show that all of these are complete intersections and conversely. Let H be the hyperplane section, then from the exact sequence

$$0 \to \mathcal{I}_X(2H) \to \mathcal{O}_{\mathbb{P}^4}(2H) \to \mathcal{O}_X(2H) \to 0,$$

we get that $h^0(X, \mathcal{O}_{\mathbb{P}^4}(2H)) - h^0(X, \mathcal{I}_X(2H)) \leq h^0(X, \mathcal{O}_X(2H)$. By Kodaira vanishing and Serre duality, we have $h^1(X, \mathcal{O}_X(2H)) = h^1(X, \mathcal{O}_X(-2H)) = 0$, so RR gives us that $h^0(X, \mathcal{O}_X(2H)) \leq 2 + \frac{1}{2}(2H)^2 = 2 + 2 \cdot 6 = 14$. But $h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2H)) = 15$. Thus $\mathcal{I}_X(2H) \geq 1$. So X is contained in at least one quadric Q. We show that it is also contained in at least one cubic not containing Q. From the same type of calculation as above, we have $h^0(X, \mathcal{O}_X(3H)) \leq 29$ while $h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3H)) = 35$, so X is contained in at least 6 linearly independent cubics. But the space of cubics that contain Q, has dimension $h^0(X, \mathcal{O}_X(H)) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(H)) = 5$. So there is at least one cubic C which does not contain Q. The inersection $Q \cap C$ has degree 6 and contains X. Thus it is equal to X. So X is a complete intersection. Conversely, a generic complete intersection of a quadric and cubic in $\mathbb{P}^4$ is K3 by the exercise in Hartshorne quoted above. We show that these again form a 19-dimensional family. We must choose a quadric from the 14-dimensional projective space of quadrics in $\mathbb{P}^4$. The linear space of cubics in $\mathbb{P}^4$ has dimension 35. But of these we choose one that is linearly independent of the quadric, so we must subtract the dimension of linear forms in $\mathbb{P}^4$ and then another one for for projectivization, i.e. 35-5-1=29. Since $\dim PGL_5 = 24$, we get a 14+29-24=19-dimensional family of such polarized K3's.

The next normal surfaces to consider are the octic K3's in $\mathbb{P}^5$. By an analysis similar to above, we get that these are generically the intersection of three quadrics and conversely. To see the dimension of this family, we see that such complete intersections are determined by choosing a 3-dimensional subspace of $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$ which has dimension 21. This is equivalent to choosing a point of $Gr(3, 21)$ which has dimension $3 \cdot 18 = 54$. Thus the family of octic polarized K3's has dimension $54 - \dim PGL_6 = 54 - 35 = 19$.

As a final example of normal K3's, we consider when the divisor C has degree 2, i.e. we have a map $\pi : X \to \mathbb{P}^2$ corresponding to a complete linear system. This gives X as a double cover of the projective plane. These curves correspond to inverse images of lines in $\mathbb{P}^2$. Restricting π to C expresses it as a double cover of such a line. By the genus formula and the fact the C is still ample since the pull back of an ample divisor by a finite morphism is ample, we get that C has genus 2. By Hurwitz's theorem we get $2g(C) - 2 = 2(-2) + \deg R$, so $\pi|_C$ is ramified over 6 distinct points. Thus the branch locus of π is a sextic curve in $\mathbb{P}^2$. Conversely, we know how to construct smooth double covers of smooth surfaces with given branch locus divisor. And we've seen that the double cover will be smooth if the branch divisor is. Thus given a smooth sextic $B \subset \mathbb{P}^2$, we can construct a double cover $\pi : X \to \mathbb{P}^2$. We know from our section on double covers that $K_X \equiv \pi^* K_{\mathbb{P}^2} + \pi^{-1}(B)$, and thus

$$2K_X \equiv 2\pi^*(K_{\mathbb{P}^2}) + 2\pi^{-1}(B) \equiv \pi^*(-6H + B) \equiv 0,$$

where this follows from the fact that $\pi^*(B) = 2\pi^{-1}(B)$. Notice that this imples that $P_2 = 1$ and that X is minimal from the genus formula. Since B has degree 6 it has genus 10, so we get $\chi(B) = 2 - 2g(B) = -18$. From our section on double covers we have that $\chi(X) = 2\chi(\mathbb{P}^2) - \chi(B) = 24$. Assuming this we get that $\chi(\mathcal{O}_X) = 2$ from Noether's formula. Thus $p_g - q = 1$, and thus $p_g \geq 1$. Since $p_g \leq P_2 = 1$, we have $p_g = 1$ and $q = 0$. From the classification theorem for $\kappa = 0$ surfaces (or a lemma above) we see that $K_X \equiv 0$. Thus X is K3. This family of K3 surfaces is thus determined by choosing a smooth sextic in $\mathbb{P}^2$. These form an open subset of $U \subset \mathbb{P}^{27}$. Thus taking the GIT quotient by PGL_3, which has dimension 8, we get this family is again a 27-8=19-dimensional family.

3 Classification of even unimodular hyperbolic lattices

We include an appendix on even hyperbolic unimodular lattices for the sake of completeness and because they come up in discussing both the cohomology of Enriques and K3 surfaces.

Lemma 3.1. *Any indefinite lattice E represents 0.*

Proof. We do this by cases. First if the rank of E is 2, then it's clear for $V := E \otimes_{\mathbb{Z}} \mathbb{Q}$, which is enough for us by rescaling. (In case this needs more explaining, it follows from the fact that we can get rid of the xy term without needs square roots. Then it's really clear). For $n = 3$, we get by CHevalley-Warning, that any quadratic form reduced mod p has a nontrivial solution. By Hensel's lemma this lifts to a solution in $\mathbb{Q}_p$ for $p \neq 2$. Thus it represents a solution in $\mathbb{Q}_p$ for every $p \neq 2$ and clearly in $\mathbb{R}$ from the indefiniteness of the signature. From a corollary of the Hasse-Minkowski theorem, we get that it in fact represents zero over $\mathbb{Q}$ (namely from the product formula for valuations). For the case $n = 4$, the same argument gets up to dealing with $\mathbb{Q}_2$. From that same corollary, if $n = 4$ and $d(E) = 1$, then we're done. Otherwise, $d(E) = -1$, which is not a square in $\mathbb{Q}_2$. But then by a general result, we get that V represents zero in $\mathbb{Q}_2$, and then in $\mathbb{Q}$ by Hasse-Minkowski. For $n \geq 5$, Meyer's theorem tells us that V represents 0. □

Lemma 3.2. *If x is an indivisible element of E there exists $y \in E$ such that $x \cdot y = 1$.*

Proof. Consider the linear form $f_x : E \to \mathbb{Z}$ given by pairing with x. Since E is unimodular, f_x isn't the zero map . Moreover, f_x is surjective, since otherwise Im f_x is a subgroup of $\mathbb{Z}$ which must be generated by a smallest element $d \geq 2$. f_x/d is then an element of $Hom(E, \mathbb{Z})$ which corresponds to pairing with some $x' \in E$, which by modularity must satisfy $x = dx'$, a contradiction to indivisibility. Thus f_x is surjective, so there is a $y \in E$ s.t. $x \cdot y = 1$. □

Lemma 3.3. *Let E be an even indefinite unimodular lattice. Then there exists a lattice F such that $E \cong U \oplus F$, where again U is the hyperbolic plane.*

Proof. From lemma 2.1, we can choose $x \in E$, $x \neq 0$, such that $x^2 = 0$. Moreover, we can choose x such that it's indivisible (i.e. it cannot be divided by a positive integer ≥ 2). From Lemma 2.2, we can pick y such that $x \cdot y = 1$. If $y^2 = 2m$, then replace y by $y - mx$ so we may assume $y^2 = 0$. The submodule G generated by x and y is thus isomorphic to U. Consider F, the orthogonal complement to G. Since U is even, unimodular, and indefinite, we get that $G \cap F = \{0\}$. For any $x \in E$, the form $f_x|_G$ corresponds to pairing with an element in G, say x_0. Then $x = x_0 + x_1$ with $x_1 \in F$. This shows that $E \cong U \oplus F$. □

Continuing this process we get the following result:

Theorem 3.4. *If E is an indefinite, even, unimodular lattice, then $E \cong U^{\oplus p} \oplus E_8^{\oplus q}$, where p, q are positive integers such that the signature of E is$8q$ and the rank is $2p + 8q$.*

4 References

[Ser] Serre, Jean-Pierre, *A Course in Arithmetic*