MMP PROBLEM SET

Here are a smattering of questions related to the lectures and to the MMP. These questions range in difficulty and some of them are open ended. I will add to them as the week goes by.

Problem 1.1. *Let* $\mathbb{P}^{14}$ *be the space of plane curves of degree* 4. *There is a natural rational map*

$$m\colon \mathbb{P}^{14} \dashrightarrow \overline{M}_3,$$

which sends a smooth plane curve C *to its isomorphism class. Let* $W \subset \mathbb{P}^{14} \times \overline{M}_3$ *be the closure of the graph and let* $p\colon W \longrightarrow \mathbb{P}^{14}$ *and* $q\colon W \longrightarrow \overline{M}_3$ *be the two projections. What does* p *blow up and what does* q *blow down? (Note that* q *is not birational).*

Problem 1.2. *Let* A *be an abelian variety. Show that* A *does not contain a rational curve.*

Problem 1.3. *Let*

$$R = \bigoplus_{m \in \mathbb{N}} R_m,$$

be a graded ring. Suppose that $R_0 = \mathbb{C}$ *so that* R *is a* $\mathbb{C}$*-algebra.*
Given $d \in \mathbb{N}$ *let*

$$R_{(d)} = \bigoplus_{m \in \mathbb{N}} R_{md},$$

the Veronese subring of R. *Suppose that* R *is an integral domain. Show that* R *is finitely generated if and only if* $R_{(d)}$ *is finitely generated. Show that this fails if* R *is not an integral domain.*

Problem 1.4. *Give an example of a smooth projective* S *containing a smooth elliptic curve* E *of negative self-intersection which cannot be contracted in the category of projective varieties.*

Problem 1.5. *Let* X *be a smooth Fano variety.*
Show that $\chi(X, \mathcal{O}_X) = 1$.

Problem 1.6. *Let* X *be a smooth Fano variety.*
Show that if $f\colon Y \longrightarrow X$ *is finite and étale then* f *is trivial.*

Problem 1.7. *Let* X *be a smooth variety of dimension* n *and let* H *be a very ample divisor.*
Show that $K_X + (n+1)H$ *is base point free and* $K_X + (n+2)H$ *is very ample.*

Problem 1.8. *Let $\pi\colon Y \longrightarrow X$ be a finite morphism of normal $\mathbb{Q}$-factorial varieties. Show that we may write*

$$K_Y + \Gamma = \pi^* K_X,$$

for some $\mathbb{Q}$-divisor Γ and identify the coefficients of Γ in terms of the ramification divisor.

Problem 1.9. *(1) Suppose that $\pi\colon Y \longrightarrow X$ is finite and Galois. Show that we may write*

$$K_Y = \pi^*(K_X + \Delta),$$

for some $\mathbb{Q}$-divisor Δ and identify the coefficients of Δ in terms of the ramification divisor.

(2) Show that

$$\inf\{\, s = \sum \frac{r_i - 1}{r_i} - 2 \,|\, r_1, r_2, \dots, r_k \in \mathbb{N}, s > 0 \,\} = \frac{1}{42}.$$

(3) Show that if C has a smooth curve of genus $g \geq 2$ then C has at most $84(g-1)$ automorphisms. (Hint: consider the quotient map $\pi\colon C \longrightarrow C/G$ where G is the automorphism group)

Problem 1.10. *Let (X, Δ) be log smooth pair. Show that if $V \subset X$ is a closed subset of codimension k and we blow up V then the log discrepancy of the exceptional divisor E is*

$$k - \sum a_i,$$

where the sum ranges over those components of Δ which contain V and a_i is the coefficient of this component in Δ.

Problem 1.11. *If S is the cone over a rational normal curve then what is the log discrepancy?*

Problem 1.12. *Let S be a normal surface.*

(1) Show that S is terminal if and only if S is smooth.
(2) Show that S is canonical if and only if S has Du Val singularities.

Problem 1.13. *Show that if (X, Δ) is not log canonical then the log discrepancy is $-\infty$.*

Problem 1.14. *Show that (X, Δ) is log canonical (respectively kawamata log terminal) if and only if there is one log resolution $\pi\colon Y \longrightarrow X$ such that the log discrepancy of every exceptional divisor is at least zero (respectively greater than zero and $\lfloor \Delta \rfloor = 0$).*

Problem 1.15. *Let X be a toric variety and let Δ be a sum of the invariant divisors. Show that (X, Δ) is log canonical.*

Definition 1.16. *Say that X is log Fano if there is a divisor Δ such that (X, Δ) is kawamata log terminal and $-(K_X + \Delta)$ is ample.*

Problem 1.17. *Let X be a normal projective variety.*
Show that the following are equivalent.

(1) *There is a divisor Δ such that (X, Δ) is kawamata log terminal and $-(K_X + \Delta)$ is ample.*
(2) *There is a divisor Δ such that (X, Δ) is kawamata log terminal, $K_X + \Delta$ is numerically trivial and Δ is big.*
(3) *There is a divisor Δ such that (X, Δ) is kawamata log terminal and $-(K_X + \Delta)$ is big.*

Problem 1.18. *Let (X, Θ) be a kawamata log terminal pair. Suppose that $-(K_X + \Theta)$ is mobile and big.*
Show that X is a Mori dream space if and only if X is log Fano. Can one drop the hypothesis that $-(K_X + \Theta)$ is mobile?